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INTERNAL WAVES IN NONHOMOGENEOUS MEDIA

BY

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CONTENTS

PREFACE

The present thesis is a part of the requirements for fulfilment of the degree of Lic.techn. (Ph.D) from the Technical University of Denmark.

The work was carried out at the Department of Fluid Mechanics, and associate professor Leif Bjørnø, Ph.D. has acted as supervisor during the study.

I am very grateful to Leif Bjørnø for his encouragement and help, the many stimulating discussions and a very pleasant collaboration.

I owe a special debt to professor N.H. Johannesen, (D.Sc) and Dr. T.N. Stevenson, the Department of the Mechanics of Fluids, University of Manchester for several important references and for their helpful comments on an earlier draft of parts of this thesis.

The help from the staff at the Department of Fluid Mechanics is very much acknowledged, and especially I want to thank my colleaque Ole Fanøe for many interesting discussions and Mrs. Krøyer for her skilful typing of the manuscript.

Lyngby, January 1976

Mogens Terp Paulsen.

ABSTRACT

This report deals with the propagation of internal gravity waves in a density stratified medium.

First a survey is presented of the mechanics at nonhomogeneous media and of some important phenomena related to and characteristic for such media. The topics discussed are mainly related to phenomena connected with wave propagation and some aspects of the stability theory for nonhomogeneous fluids.

Next, the propagation of internal waves in a medium with a two-dimensional density distribution is investigated, and special attention is paid to the case of a two-layer fluid with a harmonic perturbation of the thermocline.

Finally, the influence from viscosity on the propagation of an internal wave generated by a harmonic disturbance travelling across a bottom covered by the nonhomogeneous medium is investigated. For the case where the density distribution is exponential an analytical solution is presented.

ABSTRAKT

I denne rapport behandles udbredelsen af interne tyngdebølger i et massefylde stratificeret medium.

Først gives en oversigt over inhomogene mediers mekanik og nogle vigtige fænomener, der er knyttet til og karakteristiske for sådanne medier. De emner, der beskrives, er hovedsageligt knyttet til bølgeudbredelse i samt til visse aspekter af stabilitetsteorien for inhomogene fluider.

Dernæst undersøges udbredelsen af interne bølger i et medium med en todimensional massefyldefordeling og herunder specielt tilfældet, hvor man har et tolagsmedium med en harmonisk perturbation af termoklinen.

Til sidst undersøges viskositetsindflydelsen på udbredelsen af en intern bølge, der genereres af en harmonisk forstyrrelse, der udbreder sig hen over bunden. For det tilfælde hvor massefyldefordelingen er eksponentiel, er der fundet en analytisk løsning til problemet.

LIST OF SYMBOLS

a	Sound speed.
$\mathfrak{b}$	$\equiv B(z) + b(x,y,t)$, Buoyancy.
c	Phase velocity.
D	Water depth.
D	Diffusion coefficient (in sections 2.1 and 2.2)
d	Thermocline/interface depth.
e_{ijk}	Permutation symbol. $e_{123} = e_{231} = e_{312} = 1$, $e_{213} = e_{132} = e_{321} = -1$, $e_{ijk} = 0$ if two or more indices are equal.
g	Acceleration due to gravity.
H	Scale length.
$H()$	Heaviside unit step function.
h	Height.
i	Imaginary unit.
i,j,k	Dummy indices.
$\underline{k}$	$\equiv (\ell,m,n)$, Wave number vector.
k	Wave number.
k^*	Characteristic wave number for horizontal density distribution.
k_{eff}	$\equiv k + k^*$, Effective wave number.
k_o	Disturbance wave number.
ℓ_z	Vertical length scale.
N	Brunt-Väisälä frequency.
N_x^2	$\equiv -\frac{g}{\rho_o}\bar{\rho},x$.
N_z^2	$\equiv -\frac{g}{\rho_o}\bar{\rho},z$.
n	Mode number.
P	Polynomium.
P	Pressure amplitude function.
p	Pressure.
Q	Collection of second order terms defined in equation (3.2.13)
Q_1	Collection of second order terms defined in equation (3.2.10)

q_α	$\equiv (u,v)$, Horizontal velocity vector.
R_1	Coefficient defined in equation (4.3.7)
R_2	Coefficient defined in equation (4.3.8)
R_3	Coefficient defined in equation (4.3.9)
R_4	Coefficient defined in equation (4.3.10)
$\underline{r}$	$\equiv (x,y,z)$, Position vector.
Re	Reynolds number
Ri	Richardson number.
S	Salinity.
T	Temperature.
t	Time.
U	Mean velocity.
U	Horizontal volocity amplitude function.
u_i	$\equiv (u,v,w) \equiv (q_\alpha,w)$, Velocity vector.
u	Velocity component.
v	Velocity component.
W	Amplitude function for the vertical velocity.
We	Weber number.
w	Velocity component.
w_o	Disturbance velocity amplitude.
X	Amplitude function.
x	Horizontal coordinate.
x_i	$\equiv (x,y,z)$ Cartesian coordinates.
x^*	$\equiv k^*x$, Dimensionless length.
y	Horizontal coordinate.
Z	Amplitude function.
z	Vertical coordinate.
z_{disc}	$\equiv - d + \varepsilon^* \sin x^*$.
α	Thermal diffusivity.
α	Dummy index.
Γ	Dimensionless parameter defined in equation (2.5.11)
γ	Surface tension.
$\delta()$	Dirac's delta function.
δ_{ij}	Kronecker delta. $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & i \neq j \end{cases}$

ε	Thickness of density interface.
ε^*	Amplitude of density interface.
θ	Angle.
λ	Wave length.
λ	Second coefficient of viscosity.
μ	Viscosity.
ν	$\equiv \mu/\rho_o$, Kinematic viscosity.
ξ	Displacement perturbation.
ρ	$\equiv \bar{\rho} + \rho^*$, Density.
$\bar{\rho}$	Mean density.
ρ^*	Fluctuating density.
ρ_o	Reference density/Equilibrium density.
ρ_1	Density of upper layer in a two layer system.
ρ_2	Density of lower layer in a two layer system.
$\Delta\rho$	Density difference across interface.
σ_{xz}	Shear stress.
Φ	Viscous dissipation function (in section 2.2).
Φ	Velocity potential.
ϕ	Perturbed velocity potential.
Ω_j	Rotation vector.
ω	Angular frequency.
ω_o	Angular source frequency.
Δ	Laplacian operator.
Δ_h	Horizontal Laplacian operator.
∇	Nabla operator.
$'$	$\equiv \frac{d}{dz}$
$,j$	$\equiv \frac{\partial}{\partial x_j}$

1. INTRODUCTION

Dealing with the fluid mechanics of incompressible flow it is mostly a very good approximation to consider the density to be the same throughout the flow area. By making this approximation one has excluded the effects of buoyancy forces, which arise when a fluid, where the density varies, is subject to gravity. In areas as meteorology, physical oceanography and hydraulic engineering many important phenemena are caused by buoyancy forces, and thus it becomes crucial , when investigating phenomena in these areas, that the density variation is taken into account.

The phenomena discussed in this report all have the buoyancy force as a basic driving force, and thus the words:"nonhomogeneous medium" always designates a medium, where the density is a function of space and time.

The basic concepts of how the buoyancy forces influence the fluid motion in a nonhomogeneous fluid, is most easily understood by regarding the behaviour of a small body of homogeneous fluid in a medium with a stratification that is stable, i.e. the density is a never decreasing function of the depth. If it is now assumed that the density is not only a never decreasing, but an increasing function of the depth, then there is only one particular depth (the depth where the density of the surrounding medium is equal to the density of the fluid body considered), where the fluid body regarded can be in hydrostatic equilibrium and both above and below this depth there will be a net force on the body tending to move it back to the equilibrium depth. A more thorough discussion of this problem can be found in textbooks on mechanics of nonhomogeneous fluids, and in this connection three books especially should be mentioned. These are:"Dynamics of Nonhomogeneous Fluids" by YIH (1965), "The Dynamics of the Upper Ocean" by PHILLIPS (1969) and "Buoyancy Effects in Fluids" by TURNER (1973).

2. MECHANICS OF NONHOMOGENEOUS MEDIA

2.1 NONHOMOGENEOUS MEDIA.

The most general causes for density stratifications are variations in temperature and/or salinity, and this chapter describes some of the general physical properties of nonhomogeneous media, both those appearing in nature and those most commonly used for experimental purposes. Some terms which are frequently used in the litterature of nonhomogeneous media should first be defined: It often appears appropriate to distinguish between layers showing a comparatively small variation in density and thin layers with large density gradients separating these layers. Depending on the cause of stratification the different layers are called (ENGELUND(1972)) salinity layers separated by haloclines, temperature layers separated by thermoclines or in general: density layers separated by pyknoclines.

From a mechanical point of view, it makes no difference whether the density differences are caused by variations in temperature or in salinity, but the temperature- and salinity gradients will tend to zero due to heat conduction and salt diffusion and it is further important to note, that the time/length scales for heat conduction differs by several orders of magnitude from the scales for salt diffusion.

As an example consider heat conduction and salt diffusion in water at ca. 20 ^{o}C. At this temperature the thermal diffusivity $\alpha \simeq 1.43 \cdot 10^{-3}$ cm^2 s^{-1} and the salt diffusion coefficient $D \simeq 1.3 \cdot 10^{-5}$ cm^2 s^{-1}. A typical scale length for heat conduction is $\sqrt{\alpha t}$, where t is the time, and for salt diffusion the equivalent scale length is $\sqrt{Dt}$. A comparison between these characteristic scale lengths for different times is shown in table 1.

Time	Heat conduction	Salt diffusion
$t = 3$ min.	$\sqrt{\alpha t} \simeq 0.5$ cm.	$\sqrt{Dt} \simeq 0.5$ mm.
$t = 24$ hours	$\sqrt{\alpha t} \simeq 11$ cm.	$\sqrt{Dt} \simeq 1$ cm.

Table 1.

For a specific problem, the decision whether the influence on the density profile from heat conduction and / or salt diffusion should be included in the model must be based on an estimate of the characteristic times and lengths for the problem considered compared to the time/length scales for heat conduction and salt diffusion, but very often these effects can be neglected and, as is seen from the table, this is in particular the case when the stratification is caused by variations in salinity.

For experimental purposes there are several methods which can be used to produce a required density profile, but usually it is most convenient to use a salt solution, because it is possible in the laboratory to produce the required profile with great accuracy and because the stratification is very long lived. The stratified medium is produced by letting layers with different salt concentrations (increasing density) run in along the bottom of the tank. Immediately after the filling of the tank the density profile is about step-like, but after a very short time, molecular diffusion has reduced the steps, and the density profile is smoothed out. MOWBRAY (1967) has described and investigated several aspects of performing experiments with stratified liquids, for instance the method mentioned to produce a given stratification, a calculation of the changes of the profile because of diffusion and the optical characteristics of salt water concerning the use of schlieren and shadowgraph techniques for flow visualization.

With a solution of sodiumchloride in water it is possible to produce density variations from $\rho \simeq 1000\ \mathrm{kg\ m^{-3}}$ (pure water) to $\rho \simeq 1200$ kg m^{-3} (saturated solution) at constant temperature.

For water areas in nature the conditions are very different from place to place and from time to time, because factors as climate (temperature, wind, precipitation), water depth and the variation of the water depth (the profile of the bottom), tide, outfall from rivers etc. all have decisive influence on the density profile. To present a few examples, Fig. 1 (CARSTENS (1970)) shows the density-, salinity- and temperature profile for a fiord.

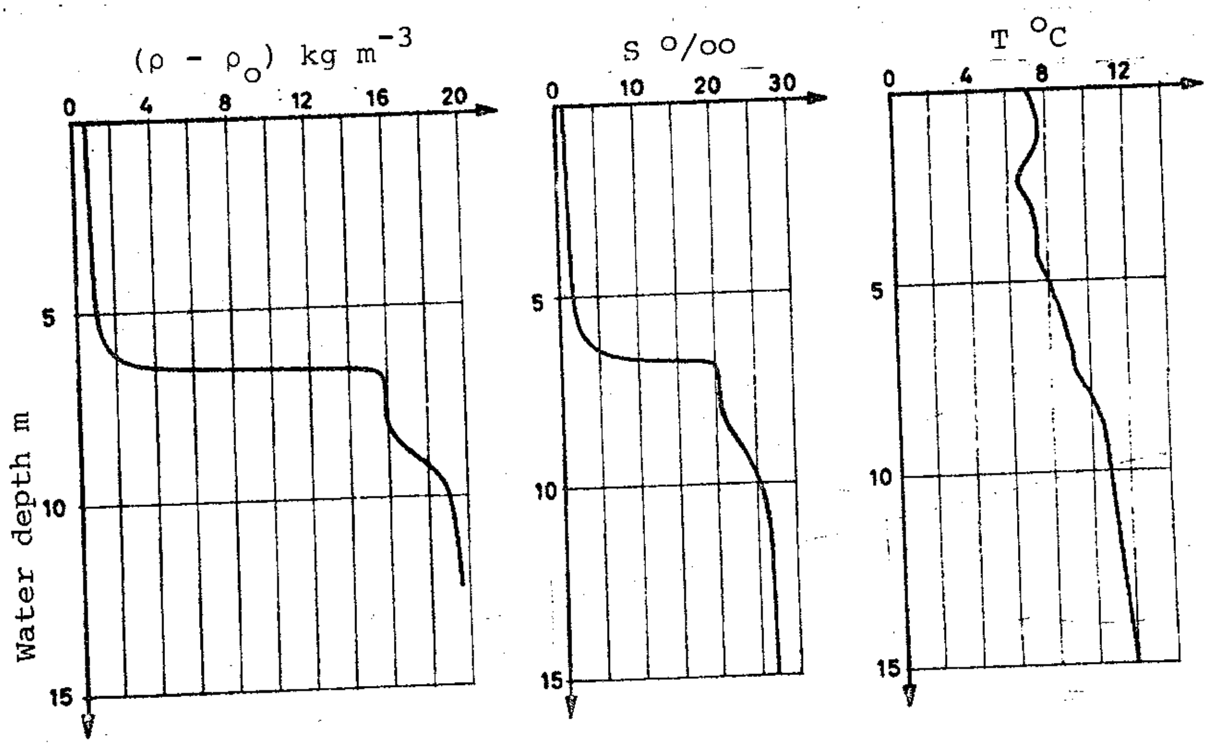

Fig. 1.

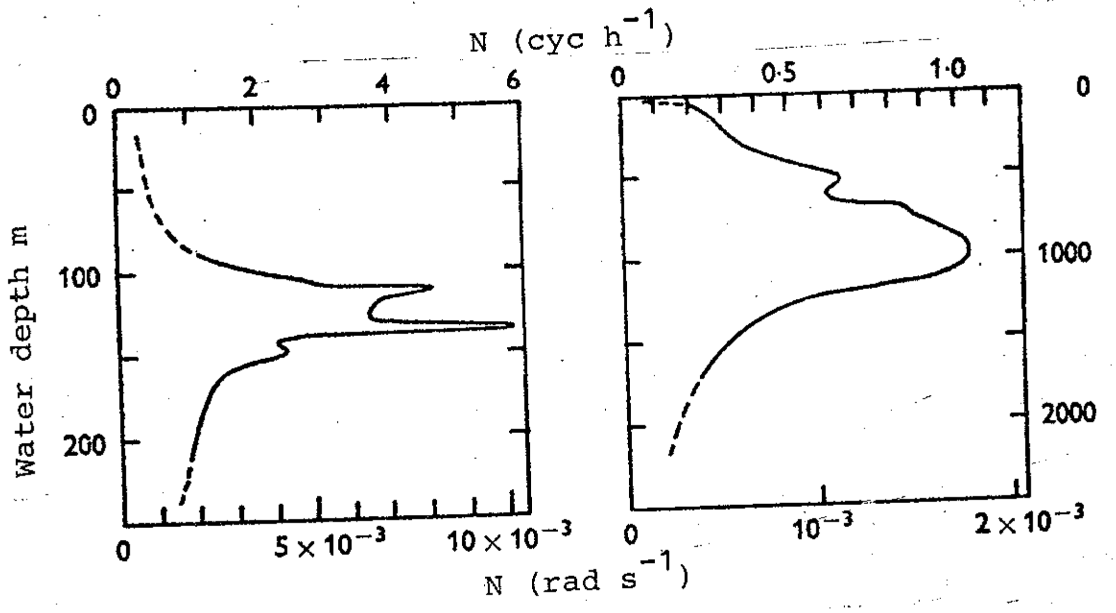

Fig. 2

Fig. 2 (after[†] PHILLIPS (1969)) shows two examples of the distribution of the Brunt-Väisälä frequency N as a function of the depth. The Brunt-Väisälä frequency is approximately defined as (see section 2.4):

$$N \simeq \sqrt{-\frac{g}{\rho_o}\frac{d\rho}{dz}} \qquad (2.1.1)$$

where g is the acceleration due to gravity, ρ is the density, ρ_o is a reference density ($\rho_o = \rho(z=0)$) and z is a coordinate that has its positive direction upwards and is zero at the surface. N has a direct physical meaning which will be discussed in section 2.4. It should be noted here that N is real when the stratification is static stable and that N is proportional to the density gradient. In the areas where the mea-

[†] Original measurements by MONTGOMERY and STROUP (1962) : Equatorial Waters and Currents at 150° W in July-August, 1952. Baltimore: The John Hopkins Press and by ISELIN (1936): A study of the circulation of the Western North Atlantic. Phys. Oceanogr. and Meteor. 6, 4.

surements for Fig. 2 were made, the density stratification is mainly caused by temperature variations and this means, that there is a thermocline at the positions where N has its maximum values on the figures. On the left curve in Fig. 2 the thermoclines are very sharp while the one to the right is more diffuse (note the different scales for the depth).

2.2 BASIC EQUATIONS.

To get a complete formulation of a fluid mechanics problem containing a certain number of unknown variables one generally must have a system of equations where the number of independent equations is equal to the number of unknown variables and in addition one must have the necessary number of boundary/initial conditions which are needed to solve the equations. The equations will usually be differential equations or integral equations (or a combination there of) and the boundary conditions are often with advantage divided into what one may call kinematic boundary conditions and dynamic boundary conditions. A kinematic boundary condition is a statement about the constraints laid upon the motion of the fluid as for example the velocity at a wall or the velocity at an interface between different layers, while a dynamic boundary condition is a statement about the stresses/forces acting on a boundary as for example the stresses on a free surface or on an interface between different layers.

The set of equations forming the basis of a description of the motion in a nonhomogeneous fluid is:

The energy equation:

$$\frac{\partial T}{\partial t} + (Tu_i)_{,i} = \alpha T_{,ii} + \Phi \qquad (2.2.1)$$

The equation of momentum balance:

$$\rho\{\frac{\partial u_i}{\partial t} + u_j u_{i,j}\} = - \rho e_{ijk}\Omega_j u_k - p_{,i} - \rho g \delta_{i3} + \{\mu(u_{i,j} + u_{j,i})\}_{,j} + \{\lambda u_{k,k}\}_{,i} \qquad (2.2.2)$$

The equation of mass conservation:

$$\frac{\partial \rho}{\partial t} + (\rho u_i)_{,i} = 0 \qquad (2.2.3)$$

The equation of conservation of disolved material:

$$\frac{\partial S}{\partial t} + (S u_i)_{,i} = D S_{,ii} \qquad (2.2.4)$$

Where $u_i \equiv (u,v,w)$ is the velocity vector, Ω_j is the rotation vector†, Φ is the viscous dissipation function, p is the static pressure, μ and λ are coefficients of viscosity and $_{,i}$ (comma) denotes partial derivatives with respect to $x_i \equiv (x,y,z)$, being cartesian coordinates. δ_{ij} is the Kronecker delta, e_{ijk} is the permutation symbol and i, j and k are dummy indices.

The equations (2.2.1) - (2.2.4) are complete in the sense, that no approximations have been introduced.

2.3 THE BOUSSINESQ APPROXIMATION.

An important example of a nonhomogeneous medium is the ocean, and it is therefore of special interest to investigate how the system of basic equations can be simplified by introducing approximations based on assumptions, which are usually valid for the ocean. One special set of approximations frequently used is known as "the Boussinesq approximation".

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†for the rotation of the earth:
$|\Omega_j| = 2\pi/12 \text{ hours} \simeq 1.45 \cdot 10^{-4}\ s^{-1}$

In order to emphasize the character and the order of magnitude of the approximations inherent in the Boussinesq approximated equations, and due to their importance and applicability, these equations shall be very briefly derived.

A more thorough discussion of the Boussinesq approximation (including its thermodynamic aspects) is given by PHILLIPS (1969). LONG (1965) has given a critical discussion of the limitations of validity of the Boussinesq approximation, in particular with respect to its use by the problem of internal solitary waves in a stratified fluid system, where special care is called for.

The approximations are introduced by comparing the real fluid system with a reference system (variables in the reference system are indicated by subscript "r"), where the salinity and the entropy are constant, and where the fluid is at rest relative to a rotating system, which usually is taken as the earth. In this reference system the equations of motion (2.2.2) are reduced to one equation

$$\frac{\partial p_r}{\partial z} + \rho_r g = 0 \tag{2.3.1}$$

and because of the assumption of constant entropy

$$dp_r = a^2 \, d\rho_r \tag{2.3.2}$$

where a is the sound speed. Combining (2.3.1) and (2.3.2) and integrating once leads to

$$\rho_r(z) = \rho_o \exp\left[-\int_o^z \frac{g}{a^2}\, dz\right] \tag{2.3.3}$$

where it is noticed, that ρ_r is characterized by a characteristic length (scale length) $H = a^2/g$. As $H \simeq 225$ km for water most often $H >> D$, where D now designates the water depth.

In the real system the most important contribution to the pressure at a given point is the weight of the fluid column above the point and therefore

$$dp = -\rho g\, dz \tag{2.3.4}$$

If, as an approximation, changes in entropy for a fluid element are neglected, then by the use of (2.3.4)

$$\frac{d\rho}{dt} = \frac{1}{a^2}\frac{dp}{dt} = -\frac{1}{a^2}\rho g w \tag{2.3.5}$$

$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u_j()_{,j}$ is the substantial derivative and $w \equiv u_3$.

Equation (2.2.3) may be rewritten on the following form

$$\frac{d\rho}{dt} + \rho u_{i,i} = 0 \tag{2.3.6}$$

which together with (2.3.5) leads to

$$\frac{d\rho}{dt} = -\rho u_{i,i} = -\frac{1}{a^2}\rho g w \tag{2.3.7}$$

or to a relation between the divergence of the total velocity field and the vertical velocity w

$$u_{i,i} = \frac{g}{a^2} w \tag{2.3.8}$$

The order of magnitude of the two terms in (2.3.8) is estimated by forming the ratio

$$\frac{\partial w/\partial z}{gw/a^2} \tag{2.3.9}$$

and introducing the quantity ℓ_z as the vertical scale length of the motion. ℓ_z is at most of the same order of magnitude as the water depth D, but generally it is smaller. (2.3.9) yields

$$\frac{\partial w/\partial z}{gw/a^2} \simeq \frac{w/\ell_z}{w/H} = \frac{H}{\ell_z} \tag{2.3.10}$$

which together with $H/D \simeq H/\ell_z >> 1$ shows that (2.3.8) in the Boussinesq approximation is replaced by

$$u_{i,i} = 0 \tag{2.3.11}$$

i.e. in this approximation the fluid motion is considered incompressible and it should be noted, that the propagation of sound waves in the ocean therefore is beyond what can be described by the Boussinesq approximated equation, since the propagation of sound waves is associated with the influence of compressibility of the fluid.

To get a simplified version of the equation of motion, equation (2.3.1) is subtracted from equation (2.2.2)

$$\rho\{\frac{\partial u_i}{\partial t} + u_j u_{i,j}\} + \rho e_{ijk}\Omega_j u_k + (p - p_r)_{,i} + (\rho - \rho_r) g \delta_{i3} = \mu(u_{i,j} + u_{j,i})_{,j} \tag{2.3.12}$$

where also equation (2.3.11) was utilized and the viscosity μ was assumed to be constant.

For the ocean $\rho/\rho_r - 1 << 1$ (usually $\rho/\rho_r - 1 < 10^{-3}$ according to PHILLIPS (1969)) is a good approximation and then the ρ in the inertia terms may be replaced by ρ_r and due to $\ell_z/H << 1$ ρ_r may be replaced by ρ_o. Summing up, (2.3.12) may now be rewritten as

$$\{\frac{\partial u_i}{\partial t} + u_j u_{i,j}\} + e_{ijk}\Omega_j u_k + \frac{p_{,i}}{\rho_o} + \frac{\rho - \rho_o}{\rho_o} g \delta_{i3} = \frac{\mu}{\rho_o}(u_{i,j} + u_{j,i})_{,j} \tag{2.3.13}$$

where p has been redefined to designate the difference between the pressure in the real system and the hydrostatic pressure in a medium at rest having the constant density ρ_o.

Equations (2.3.5), (2.3.11) and (2.3.13) are the system obtained after introduction of the Boussinesq approximation. A summing up of the most important assumptions for the approximations gives:

a) The velocities associated with the fluid motion must be small compared with the sound speed: $u_i/a \ll 1$.

b) The vertical scale length associated with the fluid motion must be small compared with the scale height for the stratification: $\ell_z/H \ll 1$.

c) The difference between the density in the real system and the density in the reference system must be small: $\rho/\rho_r - 1 \ll 1$.

The two most important consequences of the introduction of the Boussinesq approximation are 1) that the fluid motion is considered incompressible and 2) that in the equations of motion (2.3.13) the density variations are only taken into account in the buoyancy term $(\rho/\rho_o - 1)g\delta_{i3}$, while these variations are neglected in the inertia terms.

2.4 THE BRUNT-VÄISÄLÄ FREQUENCY.

The Brunt-Väisälä frequency N is a basic parameter for non-homogeneous flow and a fundamental connexion exists between the buoyancy term in (2.3.13) and N.

If the buoyancy $\hat{b}$ is defined by

$$\hat{b} \equiv - g \frac{\rho - \rho_o}{\rho_o} \qquad (2.4.1)$$

and it is assumed, that the density ρ can be divided into an average density $\bar{\rho}$ and a fluctuating density term ρ^*, i.e.

$$\rho \equiv \bar{\rho}(z) + \rho^*(x,y,t) \qquad (2.4.2)$$

where $\rho^* \ll \bar{\rho}$, and this leads to a corresponding dividing up of the buoyancy into

$$\hat{b} \equiv B(z) + b(x,y,t) \tag{2.4.3}$$

The total derivative of $\hat{b}$ with insertion of equation (2.3.5) gives

$$\frac{d\hat{b}}{dt} = \frac{g^2}{a^2} w \tag{2.4.4}$$

or, by the use of (2.4.3)

$$\frac{db}{dt} = \frac{g^2}{a^2} w - \frac{dB(z)}{dt} \tag{2.4.5}$$

The total derivative of B is

$$\frac{dB}{dt} = \frac{-g}{\rho_o} \frac{d\bar{\rho}}{dz} w \tag{2.4.6}$$

which inserted into (2.4.5) gives

$$\frac{db}{dt} = -\left[\frac{-g}{\rho_o} \frac{d\bar{\rho}}{dz} - \frac{g^2}{a^2}\right] w \tag{2.4.7}$$

or

$$\frac{db}{dt} = - N^2 w \tag{2.4.8}$$

where the abbreviation

$$N = \sqrt{\frac{-g}{\rho_o} \frac{d\bar{\rho}}{dz} - \frac{g^2}{a^2}} \tag{2.4.9}$$

is the Brunt-Väisälä frequency, which has previously been introduced. The last term under the square root in (2.4.9) is usually much smaller than the first one

$$\frac{-g}{\rho_o} \frac{d\bar{\rho}}{dz} >> \frac{g^2}{a^2} \tag{2.4.10}$$

and in most situations it is a good approximation for N, to use the following expression:

$$N = \sqrt{\frac{-g}{\rho_o}\frac{d\bar{\rho}}{dz}} \qquad (2.4.11)$$

In order to illustrate the physics of N, the free vertical oscillations of a fluid element in a stratified medium is investigated.

The vertical component of the linear, inviscid and nonrotating ($\Omega_j = 0$) version of the Boussinesq approximated equations of motion (2.3.13) is written as

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_o}\frac{\partial p}{\partial z} - g\frac{\rho^*}{\rho_o} \qquad (2.4.12)$$

and (2.2.3) gives:

$$\frac{\partial \rho^*}{\partial t} + w\frac{\partial \bar{\rho}}{\partial z} = 0 \qquad (2.4.13)$$

If a small vertical displacement ξ of a fluid element is considered, and the term comprising the pressure gradient in (2.4.12) is neglected, equations (2.4.12) and (2.4.13) give

$$\frac{\partial^2 \xi}{\partial t^2} - \frac{g}{\rho_o}\frac{\partial \bar{\rho}}{\partial z}\xi = 0 \qquad (2.4.14)$$

or

$$\frac{\partial^2 \xi}{\partial t^2} - N^2 \xi = 0 \qquad (2.4.15)$$

which shows, that the Brunt-Väisälä frequency is the natural frequency for small free oscillations of a fluid element in a stratified fluid.

The corresponding period $2\pi/N$ range in general from a few minutes in the thermocline up to several hours in the deep ocean.

2.5 STABILITY OF NONHOMOGENEOUS MEDIA.

This section deals very briefly with some aspects of the stability of flow in nonhomogeneous fluids.

Whether a nonhomogeneous fluid is static stable or not, depends on the density gradient, and the stability condition is

$$\frac{d\rho}{dz} \leq 0 \qquad (2.5.1)$$

which means that the density is a never increasing function in the positive z-direction (upwards). Another often used formulation of the same statement is, that the Brunt-Väisälä frequency N must be a real number (see equation (2.4.11)).

A simple investigation of the stability of the interface between two homogeneous fluids of different densities was made by LAMB (1945). The system consists of two semi-infinite inviscid fluids with densities ρ_1, ρ_2 and parallel uniform velocities U_1, U_2 in the positive x-direction (see Fig. 3).

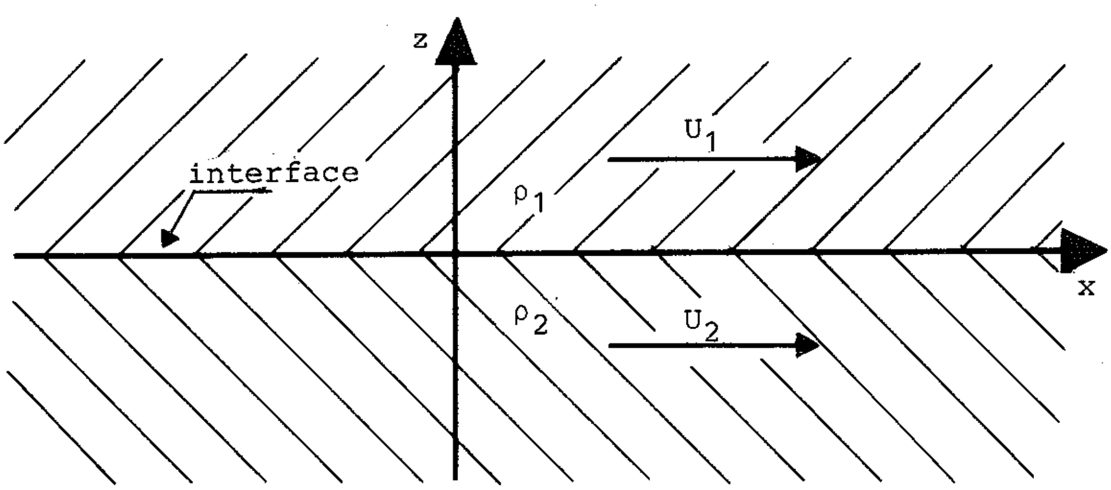

Fig. 3

The velocity potentials Φ for the upper and lower fluid, respectively, may be written as

$$\Phi_1 = - U_1 x + \phi_1 \tag{2.5.2a}$$

$$\Phi_2 = - U_2 x + \phi_2 \tag{2.5.2b}$$

where ϕ_1 and ϕ_2 are small compared to Φ_1 and Φ_2. ϕ_1 and ϕ_2 are to be determined from the equations given with one kinematic and one dynamic boundary condition to be satisfied at the interface. The kinematic boundary condition expresses, that the vertical velocity of the fluids at the interface, must be equal to the vertical velocity of the interface, and the dynamic boundary condition expresses , that the pressure must be continuous across the interface.

If ξ is the displacement perturbation of the interface from its equilibrium state $z = 0$, the linearized boundary conditions may be given by

$$\frac{\partial \xi}{\partial t} + U_1 \frac{\partial \xi}{\partial x} = - \frac{\partial \Phi_1}{\partial z} \tag{2.5.3a}$$

$$\frac{\partial \xi}{\partial t} + U_2 \frac{\partial \xi}{\partial x} = - \frac{\partial \Phi_2}{\partial z} \tag{2.5.3b}$$

and

$$\rho_1 \left[\frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x} - g\xi\right] = \rho_2 \left[\frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial x} - g\xi\right] \tag{2.5.4}$$

If it is assumed, that the perturbations ϕ_1, ϕ_2 and ξ have a harmonic time (t) and spatial (x) variation with an exponential damping in the z direction away from the interface, ϕ_1, ϕ_2 and ξ may be written as

$$\phi_1 = C_1 \, e^{i(\omega t - kx)} \cdot e^{-kz} \tag{2.5.5a}$$

$$\phi_2 = C_2 \, e^{i(\omega t - kx)} \cdot e^{kz} \tag{2.5.5b}$$

$$\xi = C_3\, e^{i(\omega t - kx)} \qquad (2.5.5c)$$

where C_1, C_2 and C_3 are constants and ω and k are the angular frequency and the wave number, respectively, for the disturbance. By inserting of (2.5.5) into the equations (2.5.3) and (2.5.4) C_1, C_2 and C_3 may be eliminated, which leads to the following equation for the angular frequency

$$\frac{\omega}{k} = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \sqrt{\frac{g}{k}\,\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} - \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)^2}\,(U_1 - U_2)^2} \qquad (2.5.6)$$

from which it is seen, that for

$$(U_1 - U_2)^2 > \frac{g}{k}\,\frac{\rho_2^2 - \rho_1^2}{\rho_1 \rho_2} \qquad (2.5.6a)$$

ω will have an imaginary part. This means, that for disturbances with sufficiently large wave numbers (small wavelengths) as given by (2.5.6), the interface will become unstable, and if $\rho_1 = \rho_2$, (2.5.6a) shows that the interface is unstable for disturbances of all wave numbers.

Systems of the kind just mentioned are very well known (f.ex. the interface between water and air), and when the theoretical results obtained are not quite in agreement with practical experiences concerning such systems it should be noted, that in the analysis stated above, a very important stabilizing effect arising from the surface tension was neglected.

An analysis of the influence of some stabilizing effects on the stability of a flowing stratified fluid was made by GRAEBEL (1960). He investigated the steady, laminar, incompressible flow of a viscous fluid. The flow took place between two parallel plates forming the angle θ with a horizontal plane and the only body force considered was the gravity. In his two-layer flow the densities of the upper and lower layer were ρ_1 and ρ_2, respectively and it was assumed, that each layer had the same viscosity μ, the same height h and that the volume mean flow was zero everywhere, see Fig. 4, which also shows the position of the coordinate system.

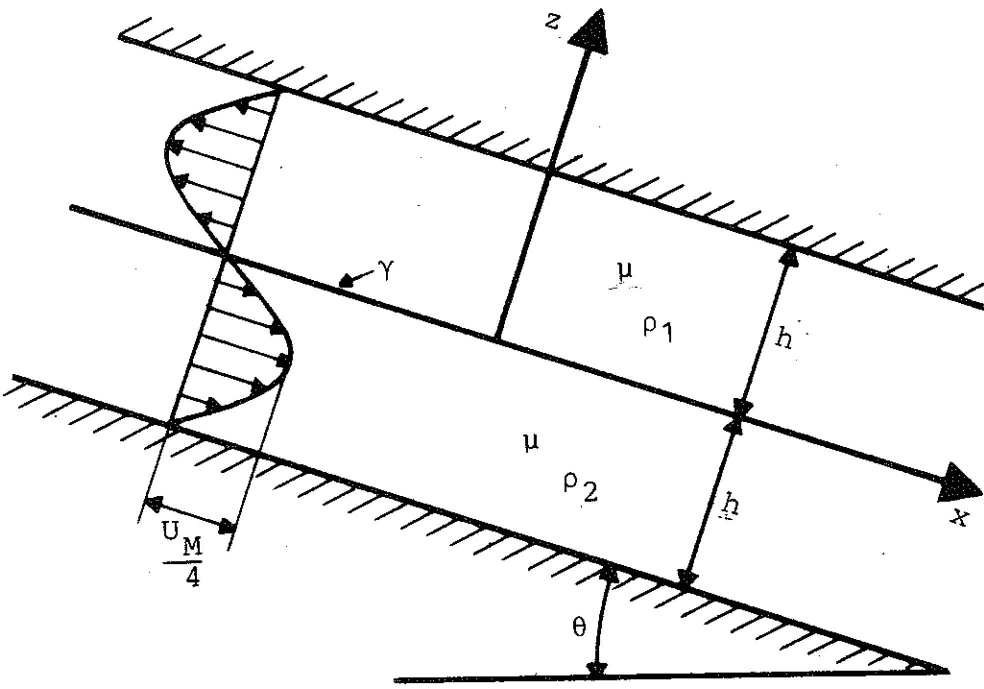

Fig. 4

Of special interest in this case is the stabilizing effect, which the surface tension γ has on the interface between the two layers.

In an investigation of the primary flow the Navier-Stokes equations show, that the velocity profile is parabolic (see Fig. 4) and that U_M , being four times the maximum velocity of the primary flow, is given by

$$U_M = \frac{\rho_2 - \rho_1}{4\mu} h^2 g \sin\theta \qquad (2.5.7)$$

The first-order equations for the velocity perturbations on dimensionless form are

$$\frac{\partial u_i}{\partial t} + U_i u_{i,x} + U_{i,z} w_i = - p_{i,x} - \frac{1}{Re_i} \Delta u_i \qquad i=1,2 \quad (2.5.8a)$$

$$\frac{\partial v_i}{\partial t} + U_i w_{i,x} = - p_{i,z} + \frac{1}{Re_i} \Delta w_i \qquad i=1,2 \quad (2.5.8b)$$

$$u_{i,x} + w_{i,z} = 0 \qquad i=1,2 \quad (2.5.9)$$

where u and w are velocity perturbations in the x and z directions, respectively, U is the velocity in the primary flow, and p is the perturbed pressure. The velocities were made dimensionless by division with U_M, the linear dimensions by division with h, the time by division with U_M/h and the pressure by division with $\rho_i U_M^2$. The subscript "i" can assume the values 1 or 2 referring to the upper and lower fluid layer, respectively. Re_i is the Reynolds number

$$Re_i = \frac{h \ U_M \ \rho_i}{\mu} \qquad i=1,2 \quad (2.5.10)$$

For the functions u, w and p it is assumed, that the disturbances they represent have a harmonic variation in the x direction and, that their amplitudes are functions of z only. Symbolically these functions may be written as:

$$f(z) \cdot \exp\{i(kx - \omega t)\} ,$$

where f(z) is the amplitude function and k and ω are the wavenumber and angular frequency of the disturbance, respectively.

The problem is not completely formulated until the boundary conditions have been specified. For the viscous fluid the no-slip condition is valid at the plates, where the velocities are zero. At the interface, the velocities and the shear stress must be continuous, while the jump in the normal stress across the interface must equal the surface tension times the curvature of the interface. The displacement ξ of the interface away from the x axis must be known. Before the boundary conditions at the interface can be formulated ξ is calculated from the equation $w = \partial\xi/\partial t$. If the boundary conditions for the normal stress at the interface are written in dimensionless form it will be seen, that the surface tension γ appears in the equations as a dimensionless parameter Γ being defined as the ratio between the Reynolds number Re and the Weber number $W = \rho h\ U_M^2/\gamma$, and is given by

$$\Gamma = \frac{\gamma}{\mu U_M} \tag{2.5.11}$$

Γ is directly proportional to the surface tension and inversely proportional to the viscosity and the maximum velocity in the primary flow.

It is assumed that the previously mentioned amplitude functions may be written as a series expansion in the z-coordinate, i.e. the solution may be written in the form $\Sigma\ a_n z^n$. If this series expansion is introduced in the differential equations, the coefficients a_n may be calculated, and the solution to the differential equations will be known. These solutions must satisfy the boundary conditions which lead to a system of homogeneous algebraic equations where the unknowns are the arbitrary constants introduced by the integration of the differential equations. The only case where a system of homogeneous algebraic equations has non-trivial solutions, is when the determinant of the coefficient matrix is zero. This last condition leads to an equation, which may be interpreted as a relation between the wave number k and the angular frequency ω, with the other variables as parameters. It is possible for a given set of parameters to calculate the wave numbers for $\mathrm{Im}(\omega) = 0$, which constitutes the stability limit. The results of such calculations are shown in Fig. 5 and in Fig. 6 (see GRAEBEL (1960)). In both figures $\rho_2/\rho_1 = 1.2$, and the curves for neutral stability ($\mathrm{Im}(\omega) = 0$) are given as a function of the disturbance wave number k and of the Reynolds number Re_1 for the flow in the upper layer with the surface tension parameter Γ as a parameter.

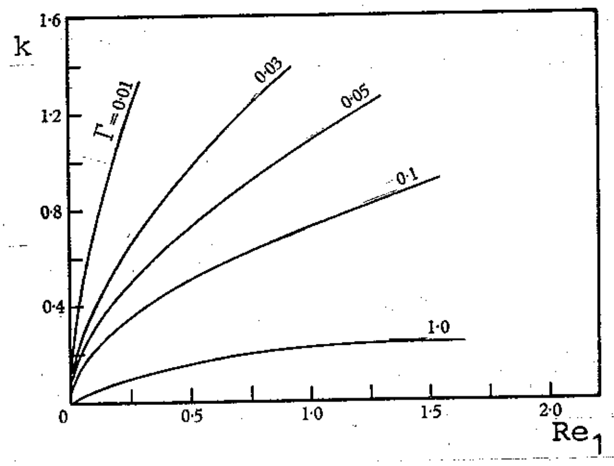

Fig. 5

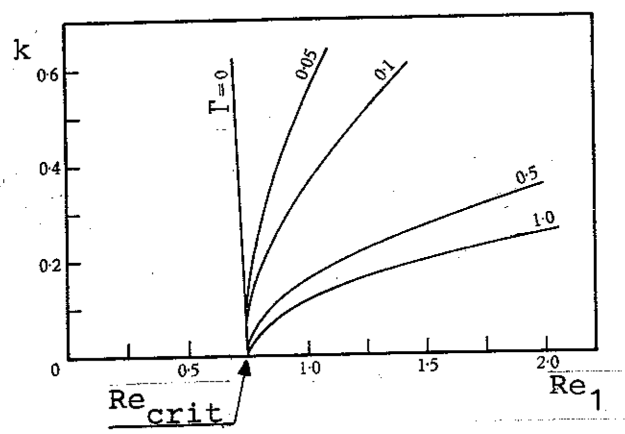

Fig. 6

The curves in Fig. 5 are for vertical flow ($\theta = 90^{\circ}$), and it may be seen, that for $\Gamma = 0$ the flow is unstable for all Reynolds numbers. The stabilizing influence of the surface tension is increasing with increasing wave number. Fig. 6 shows the stability curves for a flow with an angle of inclination $\theta = 89.5^{\circ}$, and the existence of a critical Reynolds number Re_{crit} may be seen, such that the flow is stable for all wave numbers when $Re < Re_{crit}$. The slight inclination of the curve $\Gamma = 0$ is hardly physical correct, but is probably caused by the approximations used and it is to be expected, that the curve would be vertical if more terms were included in the series expansions employed.

In the previous investigation the viscosity was the same in both layers. The case with two layers of different viscosity (and different density) was studied by KAO (1968), who investigated a two-layer flow down along an inclined plane. His system differs from the one in Fig. 4 on the following points: The wall at $z = h$ is removed, a free surface exists, and the two layers are no longer assumed to be of the same thickness. The main result of KAO's investigation is the flow stabilization obtained for $\mu_1/\mu_2 < 1$. (the layers are numbered as in Fig. 4). For $\mu_1/\mu_2 > 1$ the flow is always unstable.

The last case to be mentioned in this section is the stability of a shear flow in a nonhomogeneous medium. Investigations of this flow type were carried out by MILES (1961) and by HOWARD (1961). They considered an inviscid, incompressible flow without heat conduction. The density in the undisturbed flow was $\rho_o(z)$ and the medium was static stable, which means that the Brunt-Väisälä frequency is real. The motion of the undisturbed shear flow was described by the velocity profile $U(z)$.

Their investigations were based upon the two-dimensional equations of motion for an inviscid flow, (see (2.2.2) with $i = 1,3$, $j = 1,3$ and with $\lambda \equiv \mu \equiv \Omega_j \equiv 0$), the equation of conservation of mass (2.2.3) and the condition of incompressibility (2.3.11). The velocity perturbations u and w were introduced into these equations as $(u_1, u_3) = (U + u, w)$ and $\rho = \rho_o - \xi \cdot d\rho_o/dz$, where ξ is the particle displacement. Through a linearization the governing equations reduce to

$$\rho_o\left[\frac{\partial u}{\partial t} + Uu_{,x} + \frac{dU}{dz}w\right] = -(p - p_o)_{,x} \qquad (2.5.12a)$$

$$\rho_o\left[\frac{\partial w}{\partial t} + Uw_{,x} + N^2\xi\right] = -(p - p_o)_{,z} \qquad (2.5.12b)$$

$$\frac{\partial \xi}{\partial t} + U\xi_{,x} = w \qquad (2.5.13)$$

$$u_{,x} + w_{,z} = 0 \qquad (2.5.14)$$

It may be shown, that the dynamic instability of a static stable shear flow in a nonhomogeneous fluid can only be exponential, and ξ may therefore be given on the form

$$\xi(x,z,t) = F(z)\ e^{ik(x-ct)} \tag{2.5.15}$$

where k is the wave number and c is complex. Re(c) and Im(c) are the phase velocity and the attenuation coefficient of the disturbance, respectively. This means that the flow will be unstable for $Im(c) > 0$.

It appeared to be convenient to carry out the investigation for a layer between the heights z_1 and z_2, where z_1 and z_2 were arbitrarily chosen. For an unstable mode it may be shown that for the phase velocity Re(c) of the disturbance

$$U(z_1) < Re(c) < U(z_2) \tag{2.5.16}$$

if $dU/dz \neq 0$ in $[z_1, z_2]$.

Concerning the stability of the flow, the following theorem, as formulated by MILES (1961), can be proved: A sufficient condition for the stability of a shear flow in a nonhomogeneous medium is

$$\frac{dU(z)}{dz} \neq 0 \text{ and } Ri(z) > \frac{1}{4}$$

for $$z \in [z_1, z_2] \tag{2.5.17}$$

where Ri is the Richardson number defined by

$$Ri \equiv \frac{N^2}{\left(\frac{dU}{dz}\right)^2} \tag{2.5.18}$$

2.6 WAVE PROPAGATION.

The propagation of internal waves in a nonhomogeneous medium cannot be observed as waves on the surface, because the internal wave amplitude at the fluid surface will only be a small fraction of the maximum amplitude in the interior of the fluid. This does not imply of course that the effects of internal waves are unknown and a few examples of the effects of internal waves shall be given in the following.

Under certain circumstances during navigation there will appear what is known as "dead water". It may be felt as a reduction of the speed of the ship caused by an increased resistance. The explanation of the "dead water" phenomenon is the existence of strong internal waves, with a particle velocity close to the surface being opposite to the velocity of the ship, see Fig. 7.

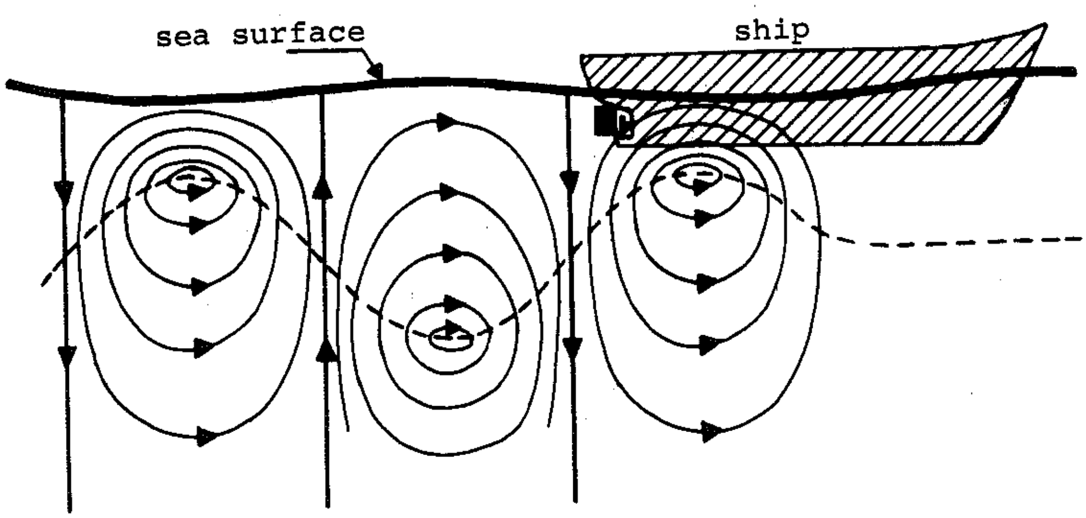

Fig. 7

Another situation where the existence of internal waves may be observed is by measurement of temperature and salinity profiles in the ocean.

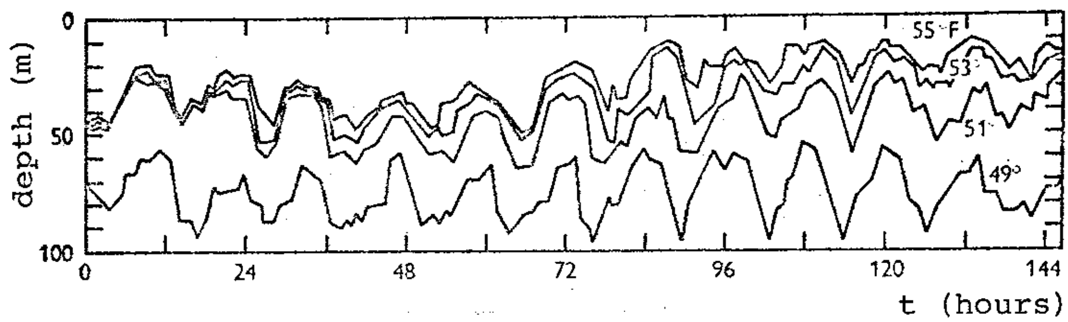

Fig. 8

Fig. 8 (after† PHILLIPS (1969)) shows measured positions of some isotherms in the ocean as a function of time. The existence of a wave motion with an amplitude of about 15 m and a dominant period very close to the period of the tide may be seen from Fig. 8. Fig. 9 (after La FOND (1962)) shows like Fig. 8 the position of isotherms but the amplitude is smaller and the frequency much higher. The waves observed are internal waves in the thermocline in shallow water.

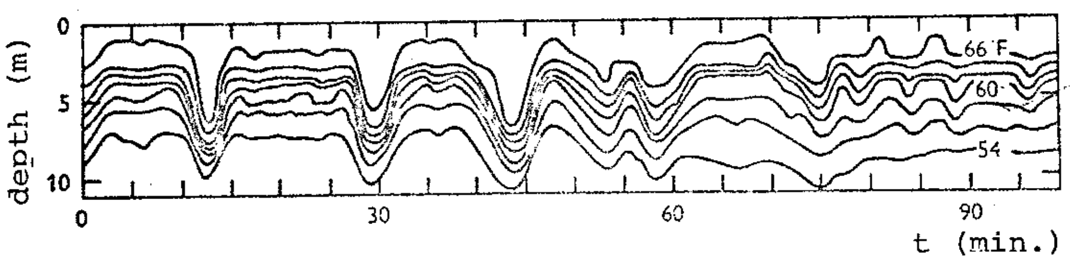

Fig. 9

The third example illustrating observable effects from internal waves concerns sound propagation in the ocean. LEE (1961) investigated the influence of an internal wave on the propagation of underwater sound radiated from a point source. Fig. 10 (after LEE (1961)) shows the sound rays in a medium

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† Original measurements published by: Reid, J.L. (1956). Observations of internal tides in October 1950. Trans. Amer. Geophys. Un. 37 , p. 278.

having a strong sound speed gradient (a thermocline) in the area between z = 30 and z = 40, and it is seen how the rays are refracted while the sound field is still being uniform.

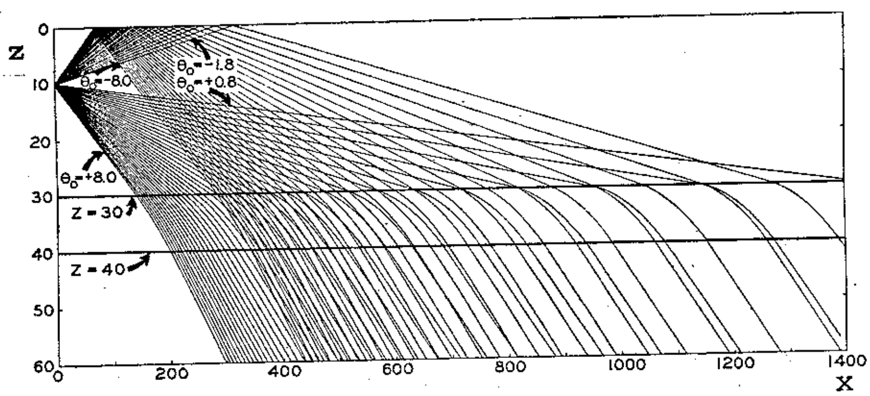

Fig. 10

The existence of an internal wave in the thermocline will lead to a refraction of the sound waves, but due to the internal wave the sound field is no longer uniform, see Fig. 11 (after LEE (1961)).

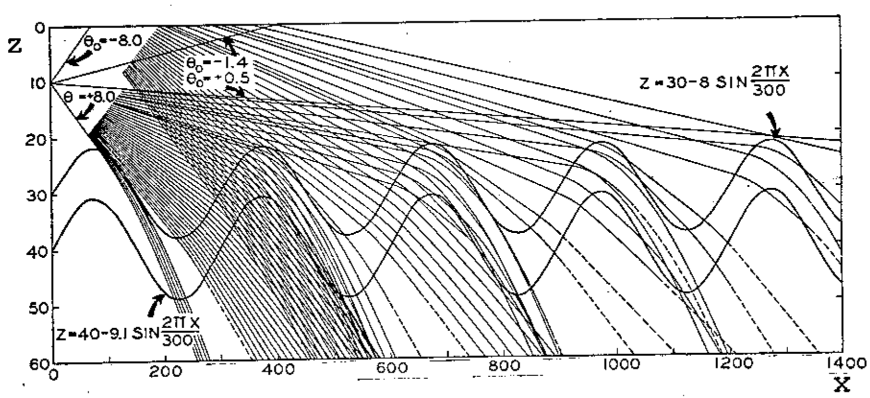

Fig. 11

There are "shadow" and "light" zones, which means zones of high sound levels and zones of low sound levels. This behaviour is confirmed by measurements in oceans carrying internal waves.

The discussion in the rest of this section will be limited to gravity waves, i.e. waves where gravity is the most important restoring force (as distinct from capillary waves, where the surface tension is the most important restoring force). The limit between capillary and gravity waves is a wave length λ of the order of magnitude $\lambda = 2\pi\sqrt{\gamma'/g}$ (see PHILLIPS (1969) and LAMB (1945)), where γ' is the surface tension divided by the density difference between the two media. Therefore, gravity waves are waves of wave length $\lambda > 2\pi\sqrt{\gamma'/g}$. For the interface between water and air at about 20 $^{\circ}$C, $2\pi\sqrt{\gamma'/g} \simeq 1{,}7$ cm.

The first subject to be discussed is a basic and important property concerning the type of the wave equation for internal gravity waves. If the propagation of internal waves in a nonhomogeneous medium is described by the linear equations of motion for incompressible and inviscid flow, it can be shown that the wave equation will be hyperbolic in a certain frequency interval, which is an important observation due to the fact that disturbances in the fluid will propagate along characteristics and these characteristics are frequently easy to determine. This observation concerning the type of the wave equation was first made by GÖRTLER (1943), but his results have obviously remained unknown to a great part of scientists, because they were later rediscovered by several authors, for instance by YIH[†] (1960) and by MOWBRAY & RARITY (1967a). Görtler's approach however is the simplest and the following account is based on his investigations.

In order to stress the principles in the method and to get as simple calculations as possible only the two-dimensional case is investigated (the extension to three dimensions is trivial). The two dimensions considered are x being the horizontal and

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[†]In the result by YIH (1960) p. 503 there is a misprint. The correct formula should be

$$\tan\gamma' = \sigma / \sqrt{-\frac{g}{\rho}\frac{d\rho}{dz} - \sigma^2}$$

z being the vertical coordinate and u and w are the velocities in the x and z directions, respectively. The equilibrium density distribution is $\rho_o(z)$ and the fluid is assumed static stable, i.e.

$$\rho_o'(z) \equiv \frac{d\rho_o(z)}{dz} \leq 0 \qquad (2.6.1)$$

The density of a fluid element which has been displaced the distance ξ (in the z direction) from its equilibrium position is assumed to be

$$\rho(z) = \rho_o(z) - \xi \cdot \rho_o'(z) \qquad (2.6.2)$$

The linearized equations of motion and the continuity equation are

$$\rho_o \frac{\partial u}{\partial t} + p_{,x} = 0 \qquad (2.6.3)$$

$$\rho_o \frac{\partial w}{\partial t} + p_{,z} + g(\rho_o - \xi\rho_o') = 0 \qquad (2.6.4)$$

$$u_{,x} + w_{,z} = 0 \quad . \qquad (2.6.5)$$

If a harmonic time variation with angular frequency ω is assumed for u, w and p of the disturbance, they may be written as

$$u = i\omega e^{i\omega t} X(x,z) \qquad (2.6.6)$$

$$w = i\omega e^{i\omega t} Z(x,z) \qquad (2.6.7)$$

$$p = p_o - g\int_{z_o}^{z} \rho_o \cdot dz + \rho_o e^{i\omega t} P(x,z) \qquad (2.6.8)$$

where X, Z and P are amplitude functions and $p_o = p(z_o)$. Introducing (2.6.6), (2.6.7) and (2.6.8) in (2.6.3), (2.6.4) and (2.6.5) gives

$$\omega^2 X - P_{,x} = 0 \qquad (2.6.9)$$

$$(\omega^2 - N^2)Z - \frac{N^2}{g} P - P_{,z} = 0 \qquad (2.6.10)$$

$$X_{,x} + Z_{,z} = 0 \qquad (2.6.11)$$

where it was utilized, that $- g \rho_o'/\rho_o \equiv N^2$. From (2.6.9), (2.6.10) and (2.6.11) differentiation easily leads to

$$\omega^2 X_{,x} - P_{,xx} = 0 \qquad (2.6.12)$$

$$\omega^2 X_{,xz} - P_{,xxz} = 0 \qquad (2.6.13)$$

$$(\omega^2 - N^2)Z_{,xx} - \frac{N^2}{g} P_{,xx} - P_{,zxx} = 0 \qquad (2.6.14)$$

$$X_{,x} = - Z_{,z} \qquad (2.6.15)$$

$$X_{,xz} = - Z_{,zz} \qquad (2.6.16)$$

where X and P may be eliminated leaving an equation for the vertical velocity amplitude function Z given by

$$(\omega^2 - N^2)Z_{,xx} + \omega^2 Z_{,zz} + \frac{N^2}{g} \omega^2 Z_{,z} = 0 \qquad (2.6.17)$$

The type of equation (2.6.17) is determined by the sign of $- (\omega^2 - N^2)\omega^2$ which shows that (2.6.17) is

$$\text{elliptic for } N^2 - \omega^2 < 0 \qquad (2.6.18)$$

$$\text{parabolic for } N^2 - \omega^2 = 0 \qquad (2.6.19)$$

$$\text{hyperbolic for } N^2 - \omega^2 > 0 \qquad (2.6.20)$$

For the hyperbolic case, the characteristics are determined from the differential equations

$$\frac{dz}{dx} = \pm \frac{\sqrt{(N^2 - \omega^2)\omega^2}}{\omega^2 - N^2} \qquad (2.6.21)$$

If θ is introduced as the angle between the horizontal plane and the characteristics, i.e. $\tan\theta = dz/dx$, equation (2.6.21) gives

$$\sin\theta = \frac{\omega}{N} \qquad (2.6.22)$$

This very important result has been confirmed by many experiments. Figure 12 (after MOWBRAY & RARITY (1967a)) shows a schlierenpicture of the wave propagation from a point source in a medium where N is nearly constant[†] and where $\omega/N = 0.615$.

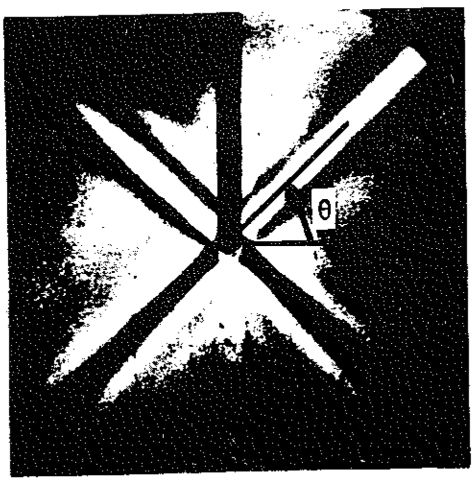


Fig. 12

[†]A medium with a linear density distribution $\rho = \rho_{z=0} - Hz$, where H is the scale height, was used. This gives $N^2 = gH/\rho$. To get constant N, an exponential density distribution $\rho = \rho_{z=0}\exp(-Hz)$ should be used, which leads to $N^2 = H \cdot g (= \text{constant})$.

If the waves are very long $\omega << N$, which is mostly the case in the ocean, they will propagate in a plane which is approximately horizontal.

The equations describing the characteristics are found by solving equation (2.6.21), which gives

$$\omega x \pm \int_{z_o}^{z} \sqrt{N^2 - \omega^2}\, dz = \text{constant} \tag{2.6.23}$$

It is seen, that for a fixed value of the frequency ω, the form of the characteristics is only a function of the relative density gradient ρ_o'/ρ_o, which is usually a function of z.

Investigating wave propagation it is often appropriate to transform the system of basic equations into one wave equation. Neglecting the rotation of the earth ($\Omega_j = 0$) and viscosity ($\mu = 0$) the Boussinesq approximated equations of motion (2.3.13) together with the incompressibility condition (2.3.11), the equation of mass conservation (2.2.3) and the assumption of splitting the density into a mean part and a fluctuating part (2.4.2) leads to

$$\frac{\partial u_i}{\partial t} + u_j u_{i,j} + \frac{1}{\rho_o} p_{,i} + \frac{\rho - \rho_o}{\rho_o} g\delta_{i3} = 0 \tag{2.6.24}$$

$$u_{i,i} = 0 \tag{2.6.25}$$

$$\frac{\partial \rho^*}{\partial t} + u_j \rho^*_{,j} + u_3 \bar{\rho}_{,3} = 0 \tag{2.6.26}$$

and these equations form the basis for the derivation of the wave equation. It appears to be convenient to distinguish between the two horizontal velocity components and the vertical velocity component and therefore $u_i \equiv (q_\alpha, w; \quad \alpha = 1,2$ is introduced. (2.6.24) then becomes

$$\frac{\partial q_\alpha}{\partial t} + \frac{1}{\rho_o} p_{,\alpha} = - u_j q_{\alpha,j} \tag{2.6.27}$$

$$\frac{\partial w}{\partial t} + \frac{1}{\rho_o} p_{,3} + \frac{\rho - \rho_o}{\rho_o} g = - u_j w_{,j} \tag{2.6.28}$$

These two equations are now reduced by a series of operations which in a symbolic way may be written as

$$\left[\frac{\partial}{\partial t} \{(2.6.27)_{,3} - (2.6.28)_{,\alpha}\}\right]_{,\alpha} \tag{2.6.29}$$

which gives

$$\frac{\partial}{\partial t} \left[\frac{\partial}{\partial t} q_{\alpha,3} + \frac{1}{\rho_o} p_{,\alpha 3} + (u_j q_{\alpha,j})_{,3} - \frac{\partial}{\partial t} w_{,\alpha} - \frac{1}{\rho_o} p_{,3\alpha} - g \frac{(\rho - \rho_o)_{,\alpha}}{\rho_o} - (u_j w_{,j})_{,\alpha}\right]_{,\alpha} = 0 \tag{2.6.30}$$

This may be rewritten as

$$\frac{\partial^2}{\partial t^2} (q_{\alpha,\alpha 3} - w_{,\alpha\alpha}) = - \frac{\partial}{\partial t} (u_j q_{\alpha,j})_{,3\alpha} + \frac{\partial}{\partial t} (u_j w_{,j})_{,\alpha\alpha} - \left(\frac{\partial}{\partial t} B(z) + \frac{\partial}{\partial t} b(x,y,t)\right)_{,\alpha\alpha} \tag{2.6.31}$$

where the definitions (2.4.1) and (2.4.3) were used. Application of equations (2.6.25) and (2.6.26) or equation (2.4.8) finally leads to

$$\frac{\partial^2}{\partial t^2} w_{,ii} + N^2 w_{,\alpha\alpha} = \frac{\partial}{\partial t} (u_j q_{\alpha,j})_{,3\alpha} - \{u_j b_{,j} - \frac{\partial}{\partial t} (u_j w_{,j})\}_{,\alpha\alpha} \tag{2.6.32}$$

In the wave equation (2.6.32) the second order terms have been collected on the right hand side of the sign of equation. These terms may in most cases (for infinitesimal disturbances and no mean shear) be neglected and the following simple linear equation is achieved

$$\frac{\partial^2}{\partial t^2} w_{,ii} + N^2(z) \cdot w_{,\alpha\alpha} = 0 \qquad (2.6.33)$$

Equation (2.6.33) is one example on a wave equation. If another starting point (i.e. other basic equations) or another method (one may write the wave equation by means of potentials or stream functions) had been chosen, a wave equation deviating from (2.6.33) will result. Examples of an alternative wave equation may be found in LAMB (1945), in GÖRTLER (1943), in YIH (1960) and in MOWBRAY & RARITY (1967a).

To solve such wave equations it is assumed that the solution is a harmonic wave, where the amplitude function is only a function of z, and an introduction of this assumption in the wave equation leads to an ordinary differential equation for the amplitude function, and this differential equation together with the boundary conditions (assuming the boundaries are not at infinity) constitute an eigenvalue problem where the wave numbers are the unknown eigenvalues. Most frequently the boundary conditions are of a type leading to an eigenvalue problem of the Sturm-Liouville type, and it is therefore possible immediately to achieve a lot of results by means of the well-known theory for Sturm-Liouville eigenvalue problems, but before discussing this further, a simple example of the use of equation (2.6.33) should be mentioned.

A phenomenon of considerable interest is the propagation of internal waves in an ocean of depth D and with a very narrow thermocline of thickness ε positioned at $z = - d$. The density jump across the thermocline is supposed to be $\Delta\rho$. The wave motion is described by (2.6.33), and it is assumed that the solution may be written as

$$w = W(z)\ e^{i(kx-\omega t)} \qquad (2.6.34)$$

where $W(z)$ is the amplitude function, k is the wave number and ω the angular frequency of the wave. Introducing (2.6.34) in

(2.6.33) gives

$$W'' + \frac{N^2 - \omega^2}{\omega^2} \cdot k^2 \; W = 0 \qquad (2.6.35)$$

Outside the thermocline it is assumed that N is zero and (2.6.35) therefore reduces to

$$W'' - k^2 \; W = 0 \qquad (2.6.36)$$

The bottom is assumed impermeable and therefore the boundary condition is $W(-D) = 0$. The boundary condition at the free surface is that the normal stress shall be equal to the atmospheric pressure, but within the Boussinesq approximation it is consistent to use the boundary condition $W(0) = 0$ (see PHILLIPS (1969)). That is,

$$W(0) = W(-D) = 0 \qquad (2.6.37)$$

The particular solution to (2.6.36) which satisfies the boundary conditions (2.6.37) is easily found to be

$$W = A \sinh(kz) \qquad , \quad 0 > z > -d \qquad (2.6.38a)$$

$$W = B \sinh\{k(z + D)\} \quad , \; -d > z > -D \qquad (2.6.38b)$$

where A and B are constants. The two parts of this solution are tied up by the conditions that W must be continuous across the thermocline. A calculation of W across the thermocline is done by integrating the complete wave equation (2.6.35) over the ε-area.

The first condition gives

$$A \sinh(-kd) = B \sinh\{k(D - d)\} \qquad (2.6.39)$$

while the second condition leads to

$$W \Big|_{-d_-}^{-d_+} = k^2 \; W(-d) \int_\varepsilon \left(\frac{N^2}{\omega^2} - 1\right) dz \qquad (2.6.40)$$

which may be written as

$$W\Big|_{-d_-}^{-d_+} = k^2\ W(-d)\left[\varepsilon - \frac{g}{\omega^2}\frac{\Delta\rho}{\rho_o}\right] \tag{2.6.41}$$

and with (2.6.38a) and (2.6.38b) as

$$A\cdot k\cdot\cosh(-kd) - B\cdot k\cdot\cosh\{k(D-d)\} = k^2\cdot A\cdot\sinh(-kd)\left[\varepsilon - \frac{g}{\omega^2}\frac{\Delta\rho}{\rho_o}\right] \tag{2.6.42}$$

The equations (2.6.39) and (2.6.42) only have non-trivial solutions for the unknown constants A and B if the determinant of the coefficient matrix is zero, and after some calculations this condition gives the dispersion relation, i.e. a relation between the wave number k and the frequency ω expressed by

$$\omega^2 = \frac{gk\Delta\rho/\rho_o}{k\varepsilon + \coth(kd) + \coth\{k(D-d)\}} \tag{2.6.43}$$

A systematic discussion of the wave equation for internal waves in stratified fluids was given by YIH (1960). His investigation was strongly based on the knowledge concerning eigenvalue problems of the Sturm-Liouville type. Some of his conclusions were:

a) For a given stratification the phase velocity c increases when the wave number k decreases (i.e. the wave length λ increases).

b) For a given wave number k the phase velocity c increases when the density gradient $d\rho/dz$ increases.

c) For a given wave number k the phase velocity c decreases when the density ρ is increased.

From the Strum-Liouville theory one may also calculate a lower and an upper bound for the phase velocity for a given system. The result is

$$\frac{-\,g\cdot D\cdot \min(d\rho/dz)}{\min(\rho)\{n^2\pi^2 + (kD)^2\}} \geq c^2 \geq \frac{-\,g\cdot D\cdot \max(d\rho/dz)}{\max(\rho)\{(n+1)^2\pi^2 + (kD)^2\}} \qquad (2.6.44)$$

where min(dρ/dz) and max(dρ/dz) are the minimum and the maximum values of the density gradient, respectively (they are both negative quantities, min(ρ) and max(ρ) are the minimum and the maximum values of the density, respectively, D is the depth and n is the mode number (i.e. there are n+1 zeros in the closed interval $0 \leq z/D \leq 1$). Yih also estimated the effect of including compressibility, and the result is that the phase velocity is decreased for the compressible flow compared to the incompressible flow case. A calculation of the phase velocities by means of a Ritz method was described and the accuracy was demonstrated in two cases, one with an exponential and one with a square root density distribution.

When studying the propagation of waves, an important method is the application of Lighthill's theory for waves in despersive media, LIGHTHILL (1967). Several authors RARITY (1967), MOWBRAY & RARITY (1967b), STEVENSON (1968) and (1969), STEVENSON & THOMAS (1969) and STEVENSON (1973) have used this method in their investigations of waves in nonhomogeneous fluids, but before discussing these investigations, the method shall be briefly outlined.

Lighthill's theory describes waves radiated from a source (oscillating or non-oscillating) moving through the fluid. It is assumed that the disturbances generated by the source are small and stable. Further, it is assumed that the waves in the otherwise source free medium are described by a linear partial differential equation with constant coefficients, given on the symbolic form

$$P\left(i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}, -i\frac{\partial}{\partial z}\right)\phi = 0 \qquad (2.6.45)$$

where P is a polynomium and φ denotes the disturbance (f.ex. represented by a velocity or a potential).

A wave of the type

$$\phi = \phi_o e^{i(\underline{k}\cdot\underline{r}-\omega t)} \qquad (2.6.46)$$

where $\underline{k} \equiv (\ell,m,n)$ is the wave number vector and $\underline{r} = (x,y,z)$ is the position vector to the observation point, is then only possible if the dispersion relation

$$P(\omega,\ell,m,n) = 0 \qquad (2.6.47)$$

is satisfied.

If the waves are supposed to be generated by a harmonic source with the angular frequency ω_o moving with a velocity $\underline{U}$ (if the source is non-oscillating $\omega_o=0$), the zero on the right hand side of (2.6.45) must be replaced by the source term

$$e^{-i\omega_o t}\cdot f(\underline{r} - \underline{U}t) \qquad (2.6.48)$$

where f is a source function.

If $F(\underline{k})$ is the Fourier transform of $f(\underline{r})$, a formal solution to the wave propagation problem described by the wave equation (2.6.45) and the source term (2.6.48) may be written as

$$\phi = \iiint_{-\infty}^{\infty} \frac{F(\underline{k})\cdot e^{i\{-\omega_o t + \underline{k}\cdot(\underline{r} - \underline{U}t)\}}}{P(\omega_o + \underline{U}\cdot\underline{k},\ell,m,n)}\, d\ell\, dm\, dn \qquad (2.6.49)$$

It is seen however that this solution is not unique because the denominator in the integrant may vanish. The physical solution of interest is the solution obtained if the source strength grows from zero to the desired strength. If this restriction is formulated mathematically, certain rules for the wave propagation and for the asymptotic behaviour of the field may be derived, see LIGHTHILL (1967). But only the most important rule shall here be discussed. This rule may be used for the determination of points of constant phase, i.e. determination of the wave form. The rule simply says, that points of constant phase are given by

$$A\,\frac{\nabla P(\omega_o + \underline{U}\cdot\underline{k},\ell,m,n)}{\underline{k}\cdot\nabla P(\omega_o + \underline{U}\cdot\underline{k},\ell,m,n)} \tag{2.6.50}$$

where A is an arbitrary constant and the nabla operator ∇ is operating on the wave number space, i.e.

$$\nabla \equiv \left(\frac{\partial}{\partial \ell}, \frac{\partial}{\partial m}, \frac{\partial}{\partial n}\right) \tag{2.6.51}$$

The direction of the wave is determined from the wave number surfaces given by

$$P(\omega_o + \underline{k}\cdot\underline{U},\underline{k}) = 0 \tag{2.6.52}$$

If arrows are drawn normal to the surface given by (2.6.52) and the directions of the arrows are such that they are normal to and are pointing towards the surface given by

$$P(\omega_o + \underline{k}\cdot\underline{U} + \delta,\underline{k}) = 0 \tag{2.6.53}$$

where δ is a small positive number (the arrows are in the direction of increasing frequency), then the arrows will indicate the direction of propagation of the wave. It should be noted that the method described is based solely on the wave equation and it is not necessary to know the solution to this equation.

In order to see how the method works in practice, a simple example will be discussed. The example is a non-oscillating source moving with the constant velocity U vertically through a medium with constant N. The wave propagation is described by equation (2.6.33) and therefore

$$P(\omega,\ell,m,n) \equiv \omega^2(\ell^2 + m^2 + n^2) - N^2(\ell^2 + m^2) \tag{2.6.54}$$

With ω_o=0, $\underline{U}$ = (0,0,U), $\underline{U}\cdot\underline{k}$ = Un and from (2.6.54)

$$\nabla P = (2\ell(U^2n^2 - N^2), 2m(U^2n^2 - N^2), 2nU^2(\ell^2 + m^2 + n^2)) \tag{2.6.55}$$

and the surface of constant phase is given from (2.6.50) as

$$\frac{A}{N^2(\ell^2 + m^2)} \left(\ell(U^2n^2 - N^2), m(U^2n^2 - N^2), nU^2(\ell^2 + m^2 + 2n^2)\right) \tag{2.6.56}$$

If Un/N is introduced as a new parameter, the surface (2.6.56) may be written as

$$r = \frac{AU}{N} \cdot \frac{(\ell - Un/N)^{3/2}}{(Un/N)^2} \tag{2.6.57a}$$

$$z = \frac{AU}{N} \cdot \left(\frac{2}{Un/N} - \frac{Un}{N}\right) \tag{2.6.57b}$$

where $r^2 = x^2+y^2$. Figure 13 (according to LIGHTHILL (1967)) shows the theoretical wave form given by equation (2.6.57)

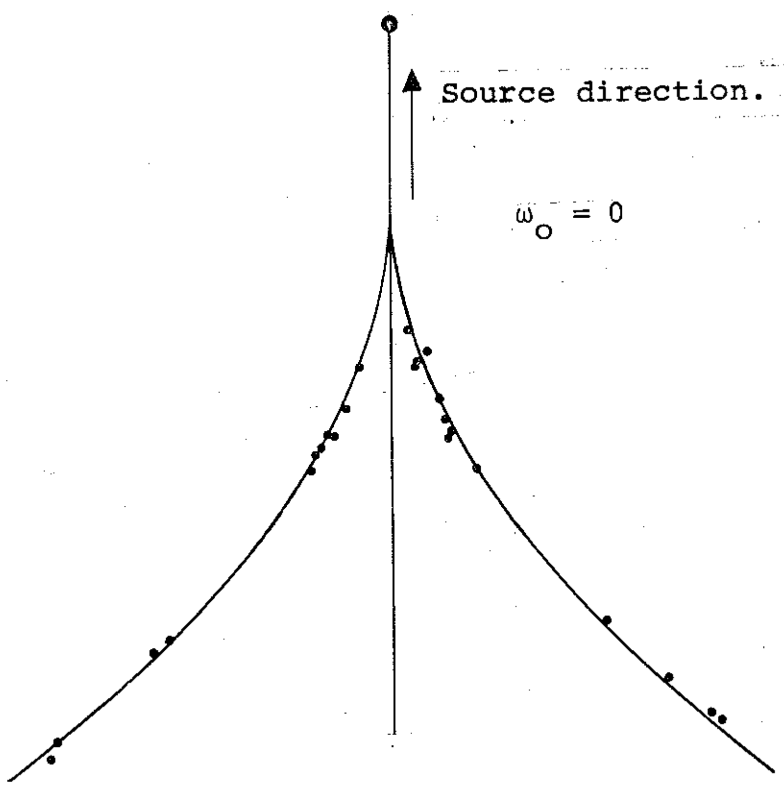


Fig. 13

The dots on the curve in Fig. 13 are experimental results measured from the schlieren picture in Fig. 14. (after LIGHTHILL (1967), photo by MOWBRAY).

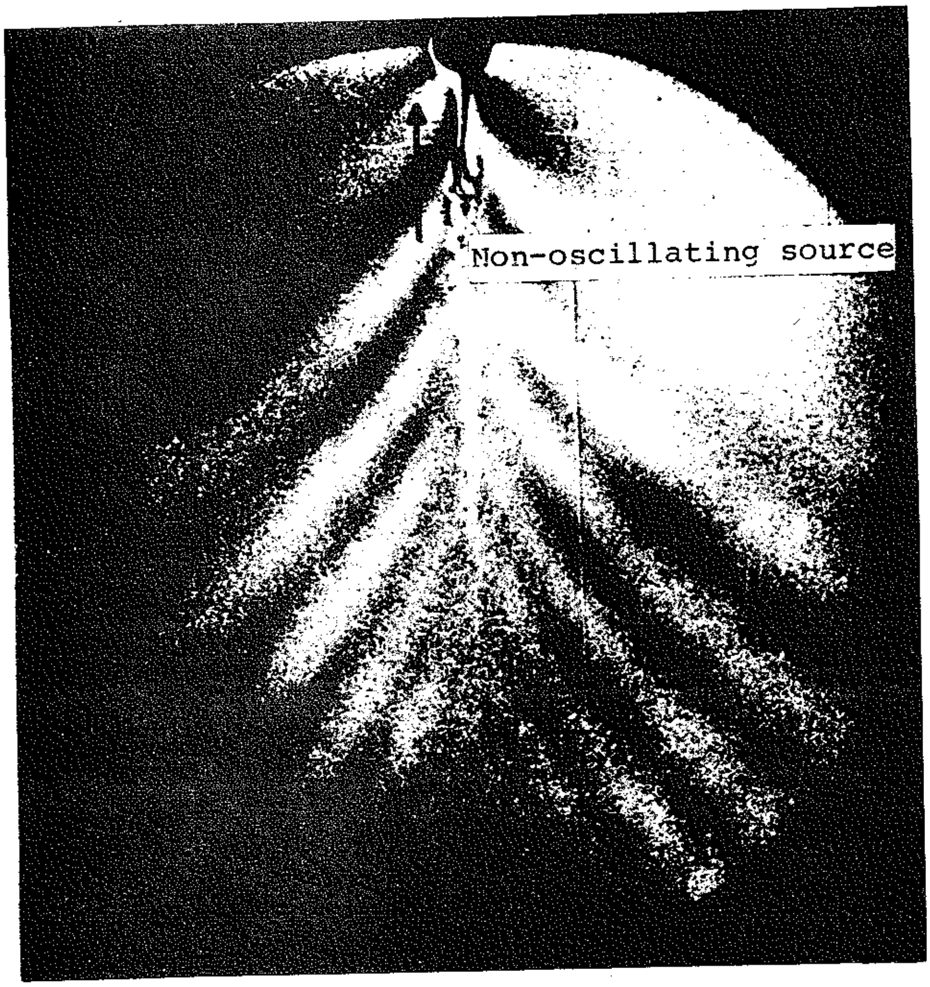


Fig. 14

The wave pattern in Fig. 14 was generated by a sphere (diameter 2.54 cm) moving vertically with a velocity of 1.02 cm s^{-1} through a brine of a density decreasing with the height by 200 kg m^{-4}.

A more comprehensive investigation of the wave pattern from a non-oscillating sphere moving vertically with a constant velocity in a nonhomogeneous medium with a linear density profile was made by MOWBRAY & RARITY (1967b), and the agreement between their experimental results and the results from Lighthill's theory is excellent.

As a continuation of the previously mentioned case of a sphere moving vertically the disturbance moving along a line of arbitrary direction has been investigated. RARITY (1967) investigated the case of a non-oscillating small disturbance (a sphere) moving with a constant velocity and STEVENSON (1968) studied the wave pattern from a non-oscillating cylinder moving with a constant velocity U. The investigation by RARITY (1967) is a very extensive analysis of the wave number surfaces (2.6.52) as a function of the angle θ between the line the disturbance is moving along and the horizontal plane; among other things he investigated the question of in which areas of the fluid the waves will propagate. STEVENSON (1968) has calculated curves of constant phase for different angles θ, and the calculations are verified experimentally by comparison with schlieren pictures. Figure 15 (after STEVENSON (1968)) shows curves of constant phase for $\theta = 45^{o}$ together with experimental results. (•: $U = 0.308$ cm s^{-1}; x: $U = 0.314$ cm s^{-1}; o: $U = 0.325$ cm s^{-1}; The diameter of the cylinder = 0.94 cm)

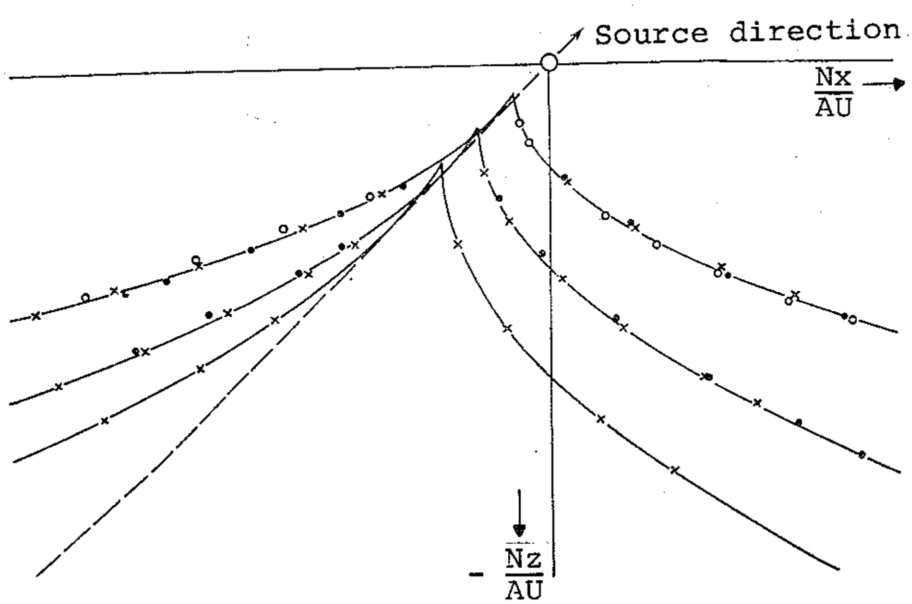


Fig. 15

The schlieren picture for U = 0.32 cm s^{-1} belonging to Fig. 15 is shown in Fig. 16 (after STEVENSON (1968)).

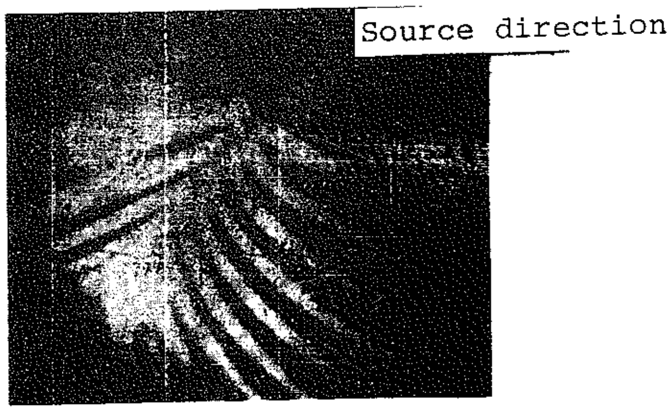


Fig. 16

The case of an oscillating moving source was investigated by STEVENSON (1969) and STEVENSON & THOMAS (1969). In the first reference the wave pattern from a source oscillating with the frequency ω_o and moving vertically with the constant velocity U was investigated. Curves of constant phase are shown in Fig. 17 (after STEVENSON (1969)) where the dashed line is for the non-oscillating source (ω_o = 0) while the full lines are for the oscillating source. In Fig. 17a ω_o/N = 0.4 , in Fig. 17b ω_o/N = 1.0 and in Fig. 17c ω_o/N = 1.4. The arrows in Fig. 17 indicate the direction and the size of the phase velocity relative to the source velocity.

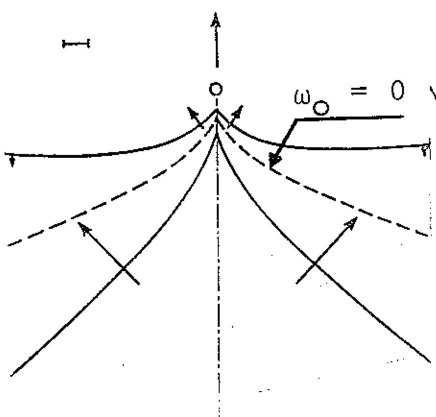


Fig. 17a

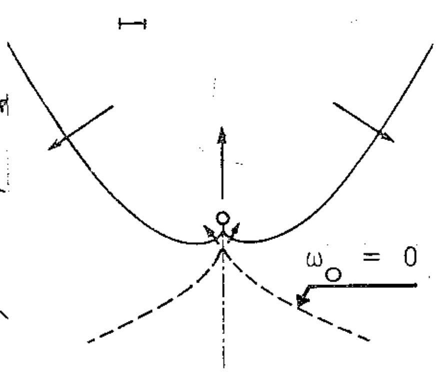


Fig. 17b

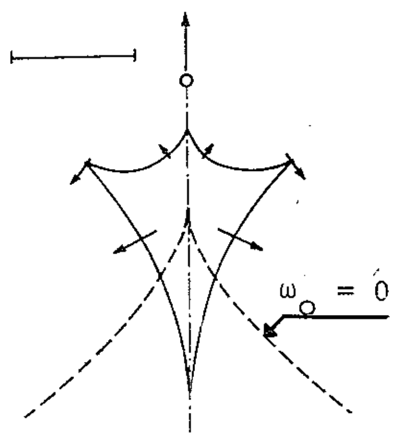


Fig. 17c

STEVENSON & THOMAS (1969) considered an oscillating cylinder moving in an arbitrary direction. An example of a schlieren picture and curves of constant phase are shown in Fig. 18 (after STEVENSON & THOMAS (1969)) and the parameters are $\theta = 45^{o}$, $U = 0.35$ cm s^{-1}, $\omega_o/N = 0.707$ with the diameter of the cylinder being 0.24 cm. Again the dashed lines are for $\omega_o = 0$.

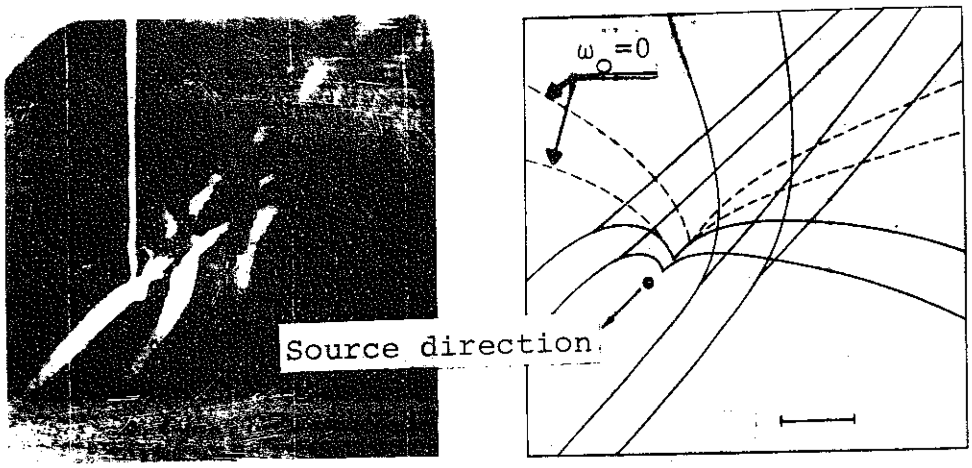


Fig. 18

The more general case where the oscillating source is moved along a curved path was investigated by STEVENSON (1973). The source speed does not have to be constant. Also the case where the source is impulsively moved a distance has been investigated.

Until now only the propagation of inviscid internal waves has been discussed. This chapter is therefore finished with the discussion of some investigations concerned with viscous, incompressible, internal wave propagation.

It may be seen f.ex. from Fig. 12 that the disturbances propagate in a narrow band along the characteristics. This observation led THOMAS & STEVENSON (1972) to the idea of looking for similarity solutions in a boundary layer type of approximation by perturbing the equations of motion in a coordinate system where one axis is along the characteristic with the other axis being perpendicular to it, i.e. the axes are parallel to the group velocity vector and to the phase velocity vector of the inviscid wave, respectively. THOMAS & STEVENSON (1972) investigated the waves for oblique characteristics, i.e. θ is different from 0 and $\pi/2$, while waves propagating vertically, i.e. the source frequency is close to the buoyancy frequency N, were investigated by GORDON & STEVENSON (1972). It should finally be mentioned that the effects of salt diffusion and of heat conduction was investigated by THOMAS & STEVENSON (1973), the effects of compressibility by STEVENSON, BEARON & THOMAS (1974) and the effects of the buoyancy frequency variation with altitude by GORDON, KLEMENT & STEVENSON (1975).

3. INTERNAL WAVES IN A MEDIUM WITH A TWO-DIMENSIONAL DENSITY DISTRIBUTION

3.1 INTRODUCTION

It is a common situation in lakes and in oceans, that the density varies abruptly, see f.ex. Fig. 1 and Fig. 2. Investigating the propagation of internal gravity waves in such water systems, it is a good approximation to the real system to apply a two-layer model and it is also a good approximation to assume that both layers have the same physical properties except that the densities of the two layers are different and equal to the actual values in the ocean or in the lake.

The propagation of internal waves in a two-layer system with a plane interface, i.e. with a one-dimensional density distribution, was discussed in section 2.6. The assumption of a plane interface is an approximation and in this chapter the effects which perturbations of the interface have on the propagation of internal gravity waves are investigated. When the interface is perturbed the density is no longer a function of z alone, but also of the horizontal coordinate x, i.e. the density distribution has become two-dimensional and this necessitates a reconsideration of the wave equation.

The fluid is assumed inviscid, and heat conduction and salt diffusion effects are not taken into account. The x,z-coordinate system is assumed to be stationary relative to the perturbations of the interface.

3.2 THE WAVE EQUATION

The wave equations in chapter 2 were derived under the assumption that the density distribution was a function of z alone but for the system under consideration in this chapter this assumption is no longer valid and a new wave equation has to be rederived. The assumptions still valid are that the motion of the fluid is described by the Boussinesq approximated equations of motion, the fluid is inviscid ($\mu = 0$) and non-rotating ($\Omega_j = 0$). It should be noted that the consequences of the Boussinesq approximation are that the variations in the fluid density are only taken into consideration in the buoyancy term in the equations of motion and that the fluid flow is considered incompressible. Further, it is assumed that the flow is two-dimensional.

On this basis the general governing equations are written as

$$\frac{\partial u_i}{\partial t} + u_j u_{i,j} = -p_{,i}/\rho_o + \left(g(\rho - \rho_o)/\rho_o\right)\delta_{i3} \tag{3.2.1}$$

$$u_{i,i} = 0 \tag{3.2.2}$$

$$\frac{\partial \rho}{\partial t} + (\rho u_i)_{,i} = 0 \tag{3.2.3}$$

where t is the time, $u_i = (u,v,w)$ is the velocity vector ($v = 0$), ,(comma as subscript) denotes partial differentiation, p is the difference between the actual pressure and the hydrostatic pressure, ρ_o is a reference density, ρ is the local density, g is the gravitational acceleration, δ_{ij} is the Kronecker delta and i and j are dummy indices.

A rewriting of the equations (3.2.1), (3.2.2) and (3.2.3) and thereby introducing the simplifying notation stated proviously leads to

$$\frac{\partial u}{\partial t} + uu_{,x} + wu_{,z} = -p_{,x}/\rho_o \tag{3.2.4}$$

$$\frac{\partial w}{\partial t} + uw_{,x} + ww_{,z} = -p_{,z}/\rho_o - g(\rho - \rho_o)/\rho_o \tag{3.2.5}$$

$$u_{,x} + w_{,z} = 0 \tag{3.2.6}$$

$$\frac{\partial \rho}{\partial t} + u\rho_{,x} + w\rho_{,z} = 0 \tag{3.2.7}$$

After a crossdifferentiation of (3.2.4) and (3.2.5) the pressure terms may be eliminated by subtracting one equation from the other, and after a differentiation with respect to x and utilizing (3.2.6) the following equation is obtained

$$\frac{\partial}{\partial t}(w_{,xx} + w_{,zz}) + g\left((\rho - \rho_o)/\rho_o\right)_{,xx} = (uu_{,x})_{,zx} + (wu_{,z})_{,zx} - (uw_{,x})_{,xx} - (ww_{,z})_{,xx} \quad (3.2.8)$$

where the terms of second order are collected on the right hand side of the sign of equation. The time derivative of equation (3.2.8) gives

$$\frac{\partial^2}{\partial t^2}\Delta w + g\frac{\partial}{\partial t}\Delta_h\left(\frac{\rho - \rho_o}{\rho_o}\right) = Q_1 \quad (3.2.9)$$

where Δ and Δ_h are the Laplacian and horizontal Laplacian operators respectively. Q_1 contains the second order terms and is defined by

$$Q_1 \equiv \frac{\partial}{\partial t}\left[(uu_{,x})_{,zx} + (wu_{,z})_{,zx} - (uw_{,x})_{,xx} - (ww_{,z})_{,xx}\right] \quad (3.2.10)$$

The time derivative of the density field in equation (3.2.9) may be replaced by terms containing products of velocities and space derivatives of the density by using equation (3.2.7). Again it is appropriate to assume that the density field may be written as a sum of a mean density $\bar{\rho}(x,z)$ and a fluctuating density term $\rho^*(x,z,t)$, i.e.

$$\rho(x,z,t) \equiv \bar{\rho}(x,z) + \rho^*(x,z,t) \quad (3.2.11)$$

which together with (3.2.7) is introduced in (3.2.9) leading to

$$\frac{\partial^2}{\partial t^2}\Delta w - \frac{g}{\rho_o}\Delta_h(u\bar{\rho}_{,x} + w\bar{\rho}_{,z}) = Q \quad (3.2.12)$$

where Q containing the second order terms is defined by

$$Q \equiv Q_1 + \frac{g}{\rho_o} \Delta_h \left(u\rho^*_{,x} + w\rho^*_{,z}\right) \qquad (3.2.13)$$

The assumption that the disturbances are infinitesimal and that there is no mean shear, makes the terms contained in Q negligible, and therefore the governing equation finally becomes

$$\frac{\partial^2}{\partial t^2} \Delta w - \frac{g}{\rho_o} \Delta_h (u\bar{\rho}_{,x} + w\bar{\rho}_{,z}) = 0 \qquad (3.2.14)$$

If desired, the u-term in (3.2.14) may be eliminated by means of (3.2.6), and a wave equation in w only is obtained. It may be seen, that if the density is a function of z only (3.2.14) becomes the well-known one-dimensional wave equation (2.6.33) for internal waves. An alternative derivation of equation (3.2.14) is shown in Appendix 1.

As first pointed out by GÖRTLER (1943) the wave equation for internal waves in a fluid with a one-dimensional density distribution, is of the hyperbolic type when the density gradient is non-zero and when $\omega < N$, where a harmonic time variation with angular frequency ω is assumed and where N is the Brunt-Väisälä frequency. (See section 2.6 for the details).

The type of equation (3.2.14) is determined from the coefficients of the highest (second) order derivatives of w. Eliminating the u-term by means of equation (3.2.6) and assuming harmonic time variation ($\exp(-i\omega t)$) equation (3.2.14) may be rewritten as

$$(\omega^2 + \frac{g}{\rho_o} \bar{\rho}_{,z}) w_{,xx} + \omega^2 w_{,zz} - \frac{g}{\rho_o} \bar{\rho}_{,x} w_{,xz} + \text{terms with lower order derivatives of } w = 0 \qquad (3.2.15)$$

and the type of equation (3.2.14) is now given as follows

$$\frac{1}{4}\left[\frac{g}{\rho_o}\bar{\rho}_{,x}\right]^2 - \omega^2\left(\omega^2 + \frac{g}{\rho_o}\bar{\rho}_{,z}\right) \begin{cases} > 0 \text{ hyperbolic} \\ = 0 \text{ parabolic} \\ < 0 \text{ elliptic} \end{cases} \tag{3.2.16}$$

Introducing the abbreviations

$$N_x^2 \equiv -\frac{g}{\rho_o}\bar{\rho}_{,x} \tag{3.2.17}$$

and

$$N_z^2 \equiv -\frac{g}{\rho_o}\bar{\rho}_{,z} \tag{3.2.18}$$

the roots of the expression in (3.2.16) are

$$\left.\begin{matrix} \omega_1^2 \\ \\ \omega_2^2 \end{matrix}\right\} = \frac{N_z^2 \pm \sqrt{N_z^4 + N_x^4}}{2} \tag{3.2.19}$$

where the root ω_1^2 refers to the plus sign and ω_2^2 to the minus sign. Since $\omega_2^2 < 0$ for $N_x^4 > 0$, ω_2^2 is not a physical root, because ω is presumed to be a real positive number. Accordingly the type of equation (3.2.14) is as shown in Fig. 19.

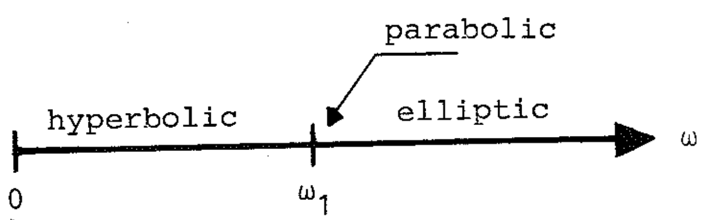


Fig. 19

and here the hyperbolic region is of interest from a physical point of view. The expression (3.2.19) should be compared with the corresponding root N^2 for the case of a one-dimensional density distribution, see (2.6.18), (2.6.19) and (2.6.20).

3.3 QUASI ONE-DIMENSIONAL DENSITY DISTRIBUTION

The influence from perturbations of the interface on the internal wave propagation is described by equation (3.2.14) using the boundary conditions for this problem. Equation (3.2.14) is a linear differential equation and therefore it is sufficient to investigate the case of a sinusoidal perturbation, because any periodic perturbation may be constructed by superposition from series of sinusoidal functions. It is however possible to go through a great part of the calculations without restricting oneself to sinusoidal perturbations, but only assume that the perturbations are periodic and that it is possible to define a characteristic wave number given by $k^* \equiv 2\pi/$(characteristic length). Further, it is appropriate to introduce a dimensionless variable defined by $x^* \equiv k^* x$, and insertion into the wave equation (3.2.14) gives

$$\frac{\partial^2}{\partial t^2}\left[k^{*2}\frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial z^2}\right]w - \frac{g}{\rho_o} k^{*2}\frac{\partial^2}{\partial x^{*2}}\left[uk^*\frac{\partial\bar{\rho}}{\partial x^*} + w\frac{\partial\bar{\rho}}{\partial z}\right] = 0 \tag{3.3.1}$$

The solutions of (3.3.1) together with (3.2.6) are assumed to be of the form

$$u = U(z)e^{ix^*}e^{i(kx^*/k^* - \omega t)} \tag{3.3.2}$$

and

$$w = W(z)e^{ix^*}e^{i(kx^*/k^* - \omega t)} \tag{3.3.3}$$

where k and ω are the wave number and the frequency of the disturbing wave motion, and U(z) and W(z) are complex amplitude functions of z only.

Insertion of (3.3.2) and (3.3.3) into (3.2.6) gives the relation

$$U = \frac{i}{k + k^*} W' \tag{3.3.4}$$

where prime denotes differentiation with respect to z. Utilizing (3.3.4) makes it possible to express the velocity u (equation (3.3.2)) by means of W as follows

$$u = \frac{i}{k + k^*} W' \, e^{i\left(\frac{k + k^*}{k^*} x^* - \omega t\right)} \tag{3.3.5}$$

Before insertion into equation (3.3.1) it is appropriate to rewrite this equation on the following form

$$\frac{\partial^2}{\partial t^2}\left[k^{*2}\frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial z^2}\right]w - \frac{g}{\rho_o} k^{*2}\left[k^* \frac{\partial^2 u}{\partial x^{*2}} \frac{\partial \bar\rho}{\partial x^*} + 2k^* \frac{\partial u}{\partial x^*}\frac{\partial^2 \bar\rho}{\partial x^{*2}} + k^* u \frac{\partial^3 \bar\rho}{\partial x^{*3}} + \frac{\partial^2 w}{\partial x^{*2}}\frac{\partial \bar\rho}{\partial z} + 2\frac{\partial w}{\partial x^*}\frac{\partial^2 \bar\rho}{\partial x^* \partial z} + w \frac{\partial^3 \bar\rho}{\partial x^{*2}\partial z}\right] = 0 \tag{3.3.6}$$

where u and w are now easily replaced by the expressions (3.3.3) and (3.3.5) which leads to

$$\begin{aligned} &- \omega^2\left[- k^{*2}\frac{(k + k^*)^2}{k^{*2}} W + W''\right] \\ &- \frac{g}{\rho_o} k^{*2}\left[- k^* \frac{i}{k + k^*} W' \frac{(k + k^*)^2}{k^{*2}} \frac{\partial \bar\rho}{\partial x^*}\right. \\ &\qquad + 2k^* \frac{i}{k + k^*} W' \, i \frac{(k + k^*)}{k^*}\frac{\partial^2 \bar\rho}{\partial x^{*2}} \\ &\qquad + k^* \frac{i}{k + k^*} W' \frac{\partial^3 \bar\rho}{\partial x^{*3}} - \frac{(k + k^*)^2}{k^{*2}} W \frac{\partial \bar\rho}{\partial z} \\ &\qquad \left. + 2i\frac{k + k^*}{k^*} W \frac{\partial^2 \bar\rho}{\partial x^* \partial z} + W \frac{\partial^3 \bar\rho}{\partial x^{*2}\partial z}\right] = 0 \end{aligned} \tag{3.3.7}$$

This equation may be reduced to the more simple one

$$W'' + W' \frac{g}{\rho_o} \frac{k^{*2}}{\omega^2} \left[- i \frac{k + k^*}{k^*} \frac{\partial \bar{\rho}}{\partial x^*} - 2 \frac{\partial^2 \bar{\rho}}{\partial x^{*2}} + i \frac{k^*}{k + k^*} \frac{\partial^3 \bar{\rho}}{\partial x^{*3}} \right]$$

$$+ W \frac{g}{\rho_o} \frac{k^{*2}}{\omega^2} \left[- \frac{\rho_o}{g} \frac{\omega^2}{k^{*2}} (k + k^*)^2 - \frac{(k + k^*)^2}{k^{*2}} \frac{\partial \bar{\rho}}{\partial z} \right.$$

$$\left. + 2i \frac{k + k^*}{k^*} \frac{\partial^2 \bar{\rho}}{\partial x^* \partial z} + \frac{\partial^3 \bar{\rho}}{\partial x^{*2} \partial z} \right] = 0 \qquad (3.3.8)$$

Again, it is important to note, that in the limit $k^* \to 0$ and $\partial(\ )/\partial x^* \equiv 0$, equation (3.3.8) reduces to

$$W'' + W \cdot k^2 \left(\frac{N^2}{\omega^2} - 1 \right) = 0 \qquad (3.3.9)$$

which is the well-known wave equation (2.6.35) for internal waves in a medium with a one-dimensional density distribution.

For a given density distribution $\bar{\rho}(x,z)$, (3.3.8) is an ordinary second order differential equation with W and z being the dependent and the independent variable, respectively. The solution to (3.3.8) will contain two arbitrary constants to be determined from the boundary conditions.

As a more specific case the system shown in Fig. 20 is discussed.

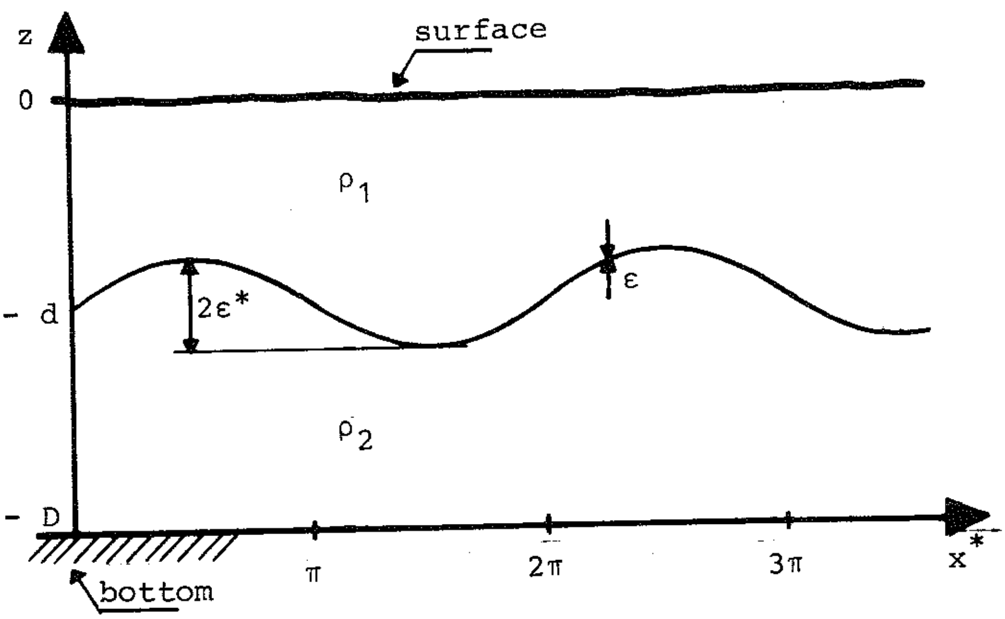


Fig. 20

The system consists of two layers with densities ρ_1 and ρ_2, respectively, and $\rho_2 - \rho_1 = \Delta\rho > 0$. Each layer is homogeneous and the interface is sinusoidal with wave number k^* and amplitude ε^*. The case of a rectilinear interface, here obtained by letting $\varepsilon^* \to 0$, was discussed in section 2.6. The surface is positioned at $z = 0$, the bottom (impermeable) at $z = -D$ and the mean interface at $z = -d$. The boundary condition at the bottom is $W(-D) = 0$ and within the Boussinesq approximation the free surface condition can be replaced by the condition $W(0) = 0$, see PHILLIPS (1969). When the solution to (3.3.8) is introduced in the boundary conditions one gets two homogeneous linear equations from which the unknown constants should be determined. In order to obtain nontrivial solutions, it is required, that the determinant of the coefficient matrix of the linear equations is zero. This determinant equation is the dispersion relation, i.e. a relation between the wave number and the frequency ω.

For the density distribution shown in Fig. 20, i.e. a two-layer system with the layer of separation being of a thickness ε, the mean density may be expressed as

$$\bar{\rho} = \rho_1 + \Delta\rho \cdot H(z_{disc} - z) \tag{3.3.10}$$

where the term $z_{disc} \equiv -d + \varepsilon^* \sin x^*$ has been introduced and $H(\ )$ is the Heaviside unit step function.

In the regions outside the density interface, the density is constant, and (3.3.8) is reduced to

$$W'' + (k + k^*)^2 W = 0 \tag{3.3.11}$$

The solutions to (3.3.11) satisfying the boundary conditions $W(0) = W(-D) = 0$ are

$$W(z) = A^+ \sinh\big((k + k^*)z\big) \qquad \text{for } z > z_{disc} + \varepsilon/2 \tag{3.3.12}$$

$$W(z) = A^- \sinh\big((k + k^*)(z + D)\big) \quad \text{for } z < z_{disc} - \varepsilon/2 \tag{3.3.13}$$

The first of the two equations necessary for the determination of the integration constants A^+ and A^- is obtained from the condition, that the vertical component of the velocity must be continuous across the density interface, i.e. W must be continuous for $z = z_{disc}$, which leads to

$$A^+ \sinh\left((k + k^*) z_{disc}\right) = A^- \sinh\left((k + k^*)(z_{disc} + D)\right) \tag{3.3.14}$$

or

$$\frac{A^+}{A^-} = \frac{\sinh\left((k + k^*)(D - d + \varepsilon^* \sin x^*)\right)}{\sinh\left((k + k^*)(- d + \varepsilon^* \sin x^*)\right)} \,, \quad 0 < x^* < 2\pi \tag{3.3.15}$$

The second equation is obtained by integration of the complete wave equation across the density interface. Before (3.3.8) can be integrated, the expressions for the derivatives of the density must be inserted. The individual terms are listed in Appendix 2. After having rearranged the terms one gets

$$\begin{aligned} W'' + W' \frac{g}{\rho_o} \frac{k^{*2}}{\omega^2} \Delta\rho \cdot \varepsilon^* \Big(& \delta()\left(- i \frac{k + k^*}{k^*} \cos x^* + 2 \sin x^*\right. \\ & \left. - i \frac{k^*}{k + k^*} \cos x^*\right) + \delta'()\left(- 2\varepsilon^* \cos^2 x^* - 3i \frac{k^*}{k + k^*}\right. \\ & \left. \times \varepsilon^* \sin x^* \cos x^*\right) + \delta''()\; i \frac{k^*}{k + k^*} \varepsilon^{*2} \cos^3 x^* \Big) \\ + W \frac{g}{\rho_o} \frac{k^{*2}}{\omega^2} \Big(& - \frac{\rho_o}{g} \frac{\omega^2}{k^{*2}} (k + k^*)^2 + \delta() \frac{(k + k^*)^2}{k^{*2}} \Delta\rho \\ & + \delta'()\left(\Delta\rho \cdot \varepsilon^* \sin x^* - 2i \frac{k + k^*}{k^*} \Delta\rho \cdot \varepsilon^* \cos x^*\right) \\ & - \delta''() \cdot \Delta\rho \cdot \varepsilon^{*2} \cos^2 x^* \Big) = 0 \end{aligned} \tag{3.3.16}$$

where δ is the Dirac delta function and $\delta() \equiv \delta(-d + \varepsilon^* \sin x^* - z)$ etc.

Equation (3.3.16) is integrated in the z-direction from a point $z_{disc-} = z_{disc} - \varepsilon/2$ just below the interface to a point $z_{disc+} = z_{disc} + \varepsilon/2$ just above the interface. The integration is performed term by term, and the fact, that W is continuous, i.e. $W(z_{disc-}) = W(z_{disc+}) = W(z_{disc})$, is utilized by taking W outside the integral sign. The details of the integration are found in Appendix 3, and the result is

$$W'(z_{disc+}) - W'(z_{disc-}) = W(z_{disc})\left((k + k^*)^2 \cdot \varepsilon - g \frac{\Delta\rho}{\rho_o} \frac{(k + k^*)^2}{\omega^2}\right) \quad (3.3.17)$$

Introducing (3.3.12) and (3.3.13) in (3.3.17) gives the second equation for the constants A^+ and A^-

$$A^+(k + k^*)\cosh\left((k + k^*)(- d + \varepsilon^* \sin x^*)\right) - A^-(k + k^*)\cosh\left((k + k^*)(D - d + \varepsilon^* \sin x^*)\right) = A^+ \sinh\left((k + k^*)(- d + \varepsilon^* \sin x^*)\right)\left((k + k^*)^2 \varepsilon - g \frac{\Delta\rho}{\rho_o} \frac{(k + k^*)^2}{\omega^2}\right) \quad (3.3.18)$$

Equation (3.3.8) may be rewritten

$$\frac{A^+}{A^-} = \cosh\{(k + k^*)(D - d + \varepsilon^* \sin x^*)\} \times \left[\cosh\{(k + k^*)(- d + \varepsilon^* \sin x^*)\} - \sinh\{(k + k^*)(- d + \varepsilon^* \sin x^*)\}\left((k + k^*)\varepsilon - g \frac{\Delta\rho}{\rho_o} \frac{k + k^*}{\omega^2}\right)\right]^{-1} \quad (3.3.19)$$

Combination of (3.3.15) and (3.3.19) leads to

$$\frac{\sinh\{(k + k^*)(D - d + \varepsilon^* \sin x^*)\}}{\sinh\{(k + k^*)(- d + \varepsilon^* \sin x^*)\}} = \cosh\{(k + k^*)(D - d + \varepsilon^* \sin x^*)\} \times \Big[\cosh\{(k + k^*)(- d + \varepsilon^* \sin x^*)\} - \sinh\{(k + k^*)(- d + \varepsilon^* \sin x^*)\}\Big((k + k^*)\varepsilon - g \frac{\Delta\rho}{\rho_o} \frac{k + k^*}{\omega^2}\Big)\Big]^{-1} \qquad (3.3.20)$$

which, after being divided by

$\cosh\{(k + k^*)(D - d + \varepsilon^* \sin x^*)\}/\sinh\{(k + k^*)(- d + \varepsilon^* \sin x^*)\}$

on both sides of equation (3.3.20) may be reduced to

$$\coth\{(k + k^*)(D - d + \varepsilon^* \sin x^*)\} = \coth\{(k + k^*)(- d + \varepsilon^* \sin x^*)\} - (k + k^*)\varepsilon - g \frac{\Delta\rho}{\rho_o} \frac{k + k^*}{\omega^2} \qquad (3.3.21)$$

Solving for ω^2 this equation finally gives the dispersion relation in an appropriate form

$$\omega^2 = \frac{g \frac{\Delta\rho}{\rho_o}(k+k^*)}{(k+k^*)\varepsilon + \coth\big((k+k^*)(D-d+\varepsilon^* \sin x^*)\big) - \coth\big((k+k^*)(-d+\varepsilon^* \sin x^*)\big)} \qquad (3.3.22)$$

By letting $k^* \to 0$ and $\varepsilon^* \to 0$ it is easily verified, that (3.3.22) becomes the well-known dispersion relation for a two-layer system with a plane surface of separation, see equation (2.6.43).

It should be noticed from equation (3.3.22), that the wave numbers k and k^* always appear as a sum, i.e. the frequency of the internal wave is unaffected by the proportion between k and k^*, and one could introduce an effective wave number k_{eff} as $k_{eff} \equiv k + k^*$ leading to

$$\omega^2 = \frac{g\,\frac{\Delta\rho}{\rho_o}\cdot k_{eff}}{k_{eff}\cdot\varepsilon+\coth\left(k_{eff}(D-d+\varepsilon^*\sin x^*)\right)-\coth\left(k_{eff}(-d+\varepsilon^*\sin x^*)\right)} \qquad (3.3.23)$$

In order to elucidate the properties of (3.3.23), some numerical results are shown in the following figures. The parameters which are the same in all figures are: $g = 9.8\ m\cdot s^{-2}$, $\rho_o = 1000\ kg\cdot m^{-3}$, $\Delta\rho = 20\ kg\cdot m^{-3}$ and $D = 500\ m$. In Fig. 21 and Fig. 22 the angular frequency ω as a function of the effective wave number k_{eff} is shown for different values of the interface thickness ε and the dimensionless distance x^*, respectively

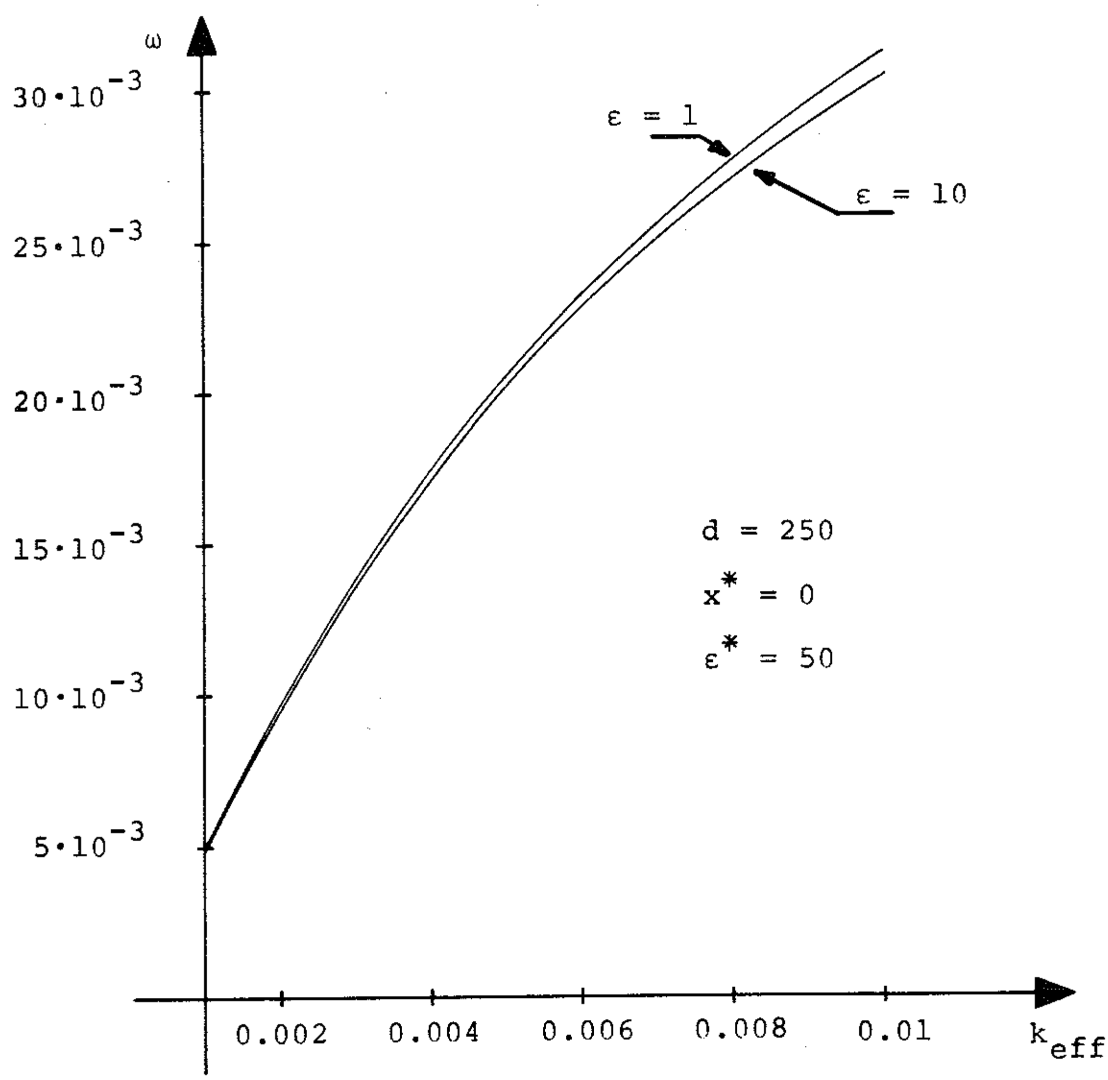
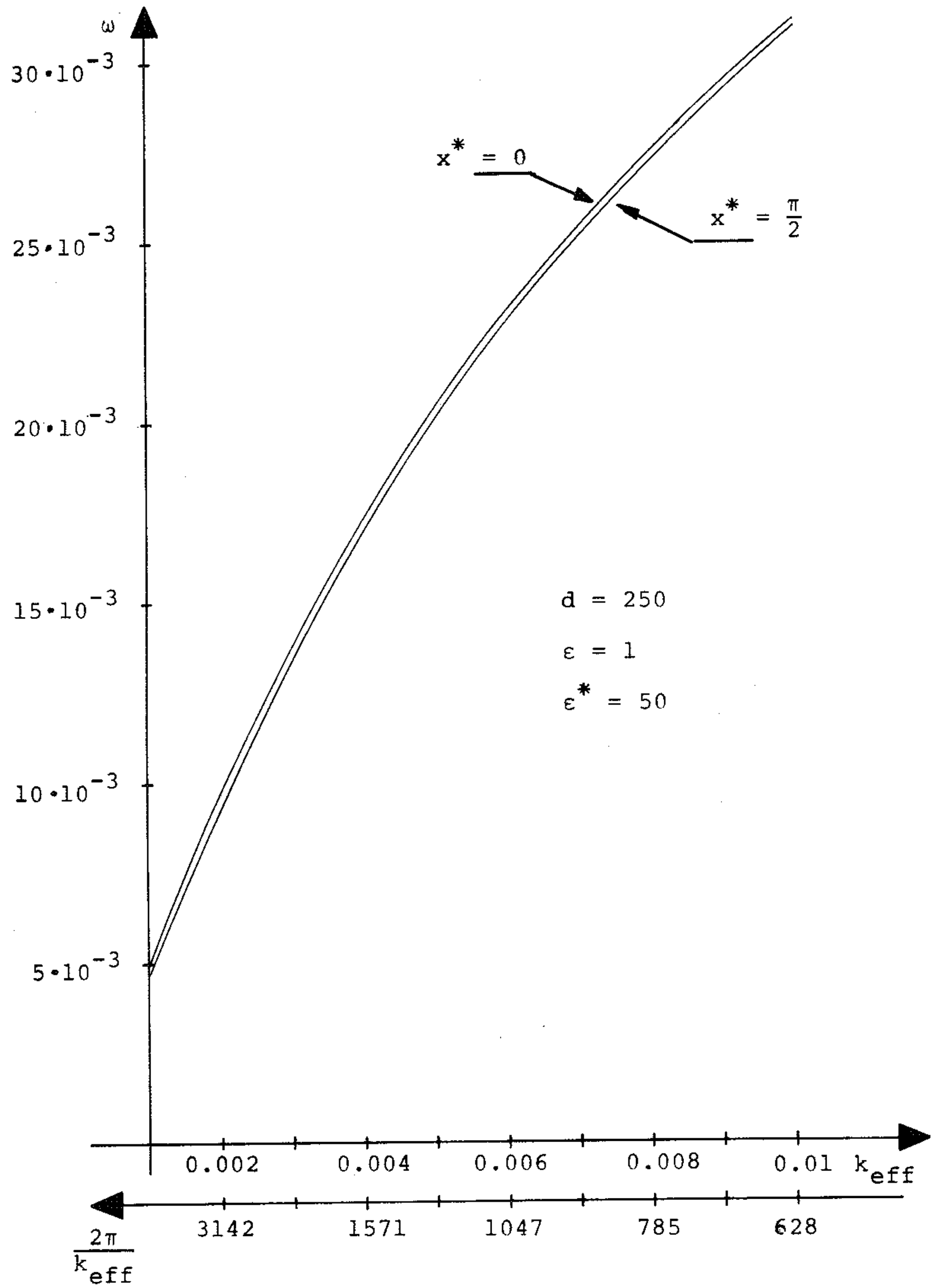


Fig. 21



ω
30·10⁻³
25·10⁻³
20·10⁻³
15·10⁻³
10·10⁻³
5·10⁻³
x* = 0
x* = π/2
d = 250
ε = 1
ε* = 50
0.002
0.004
0.006
0.008
0.01
k_eff
3142
1571
1047
785
628
2π/k_eff

Fig. 22

The variation of the angular frequency ω as a function of the dimensionless distance x^* for different values of the amplitude of the interface is shown in Fig. 23 for the case $k_{eff} = 0.001$ and in Fig. 24 for the case $k_{eff} = 0.005$. For reference, the shape of the interface is also shown in these figures.

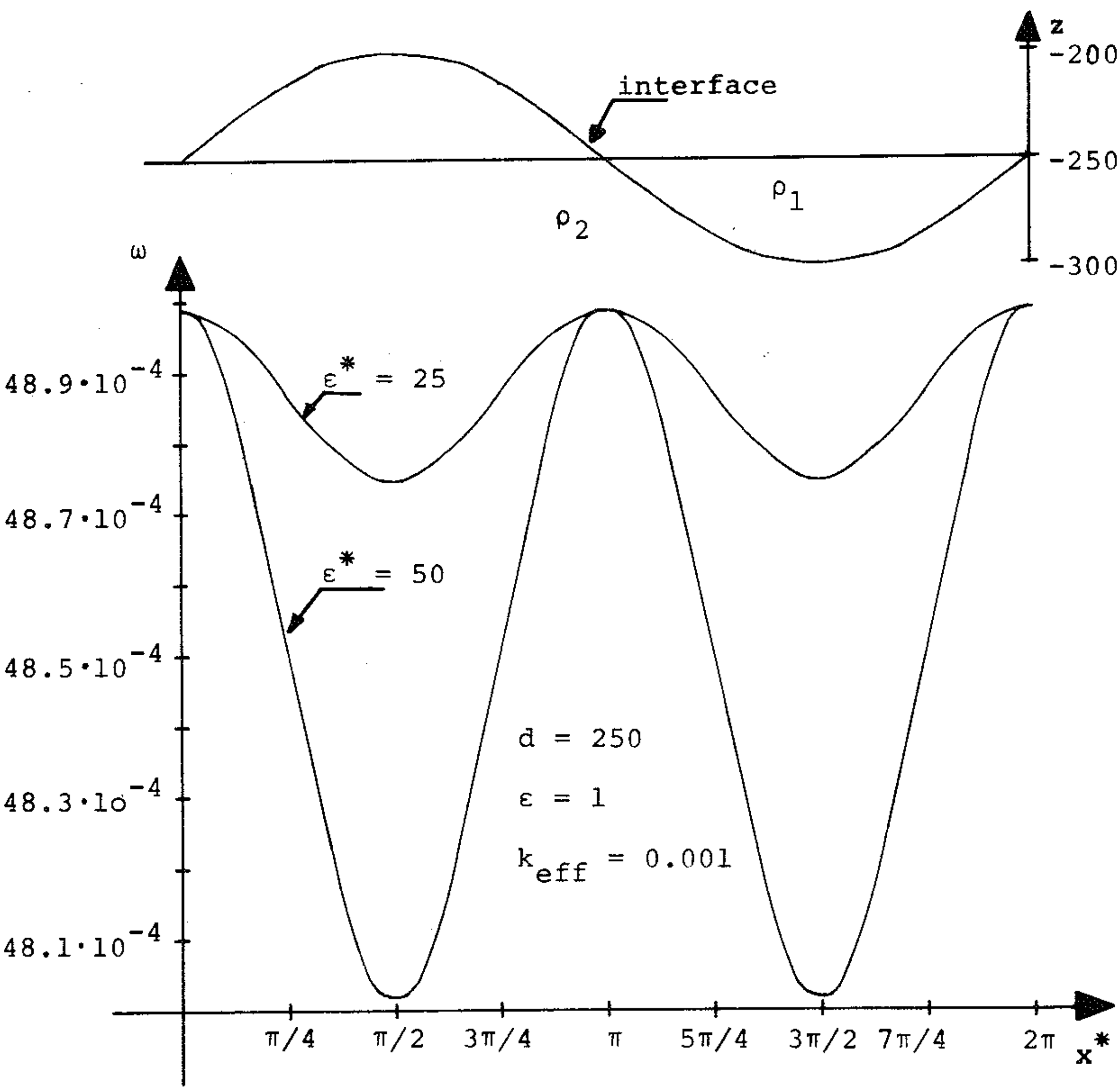


Fig. 23.

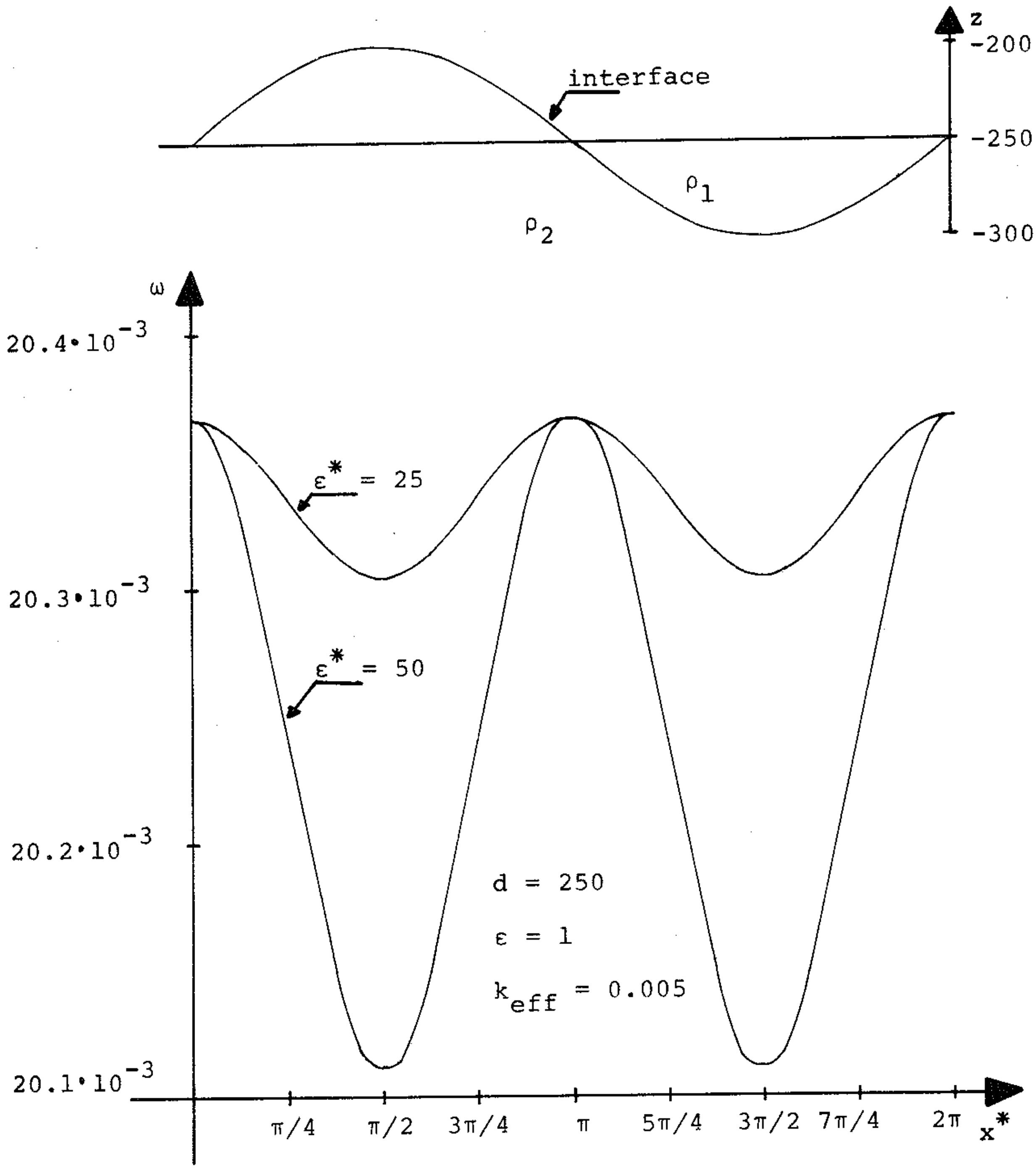


Fig. 24

In the previous figures the density interface was halfway between the bottom and the surface. In Fig. 25 ω is shown as a function of x^* for two different positions of the interface, and for two different values of the interface amplitude ε^*.

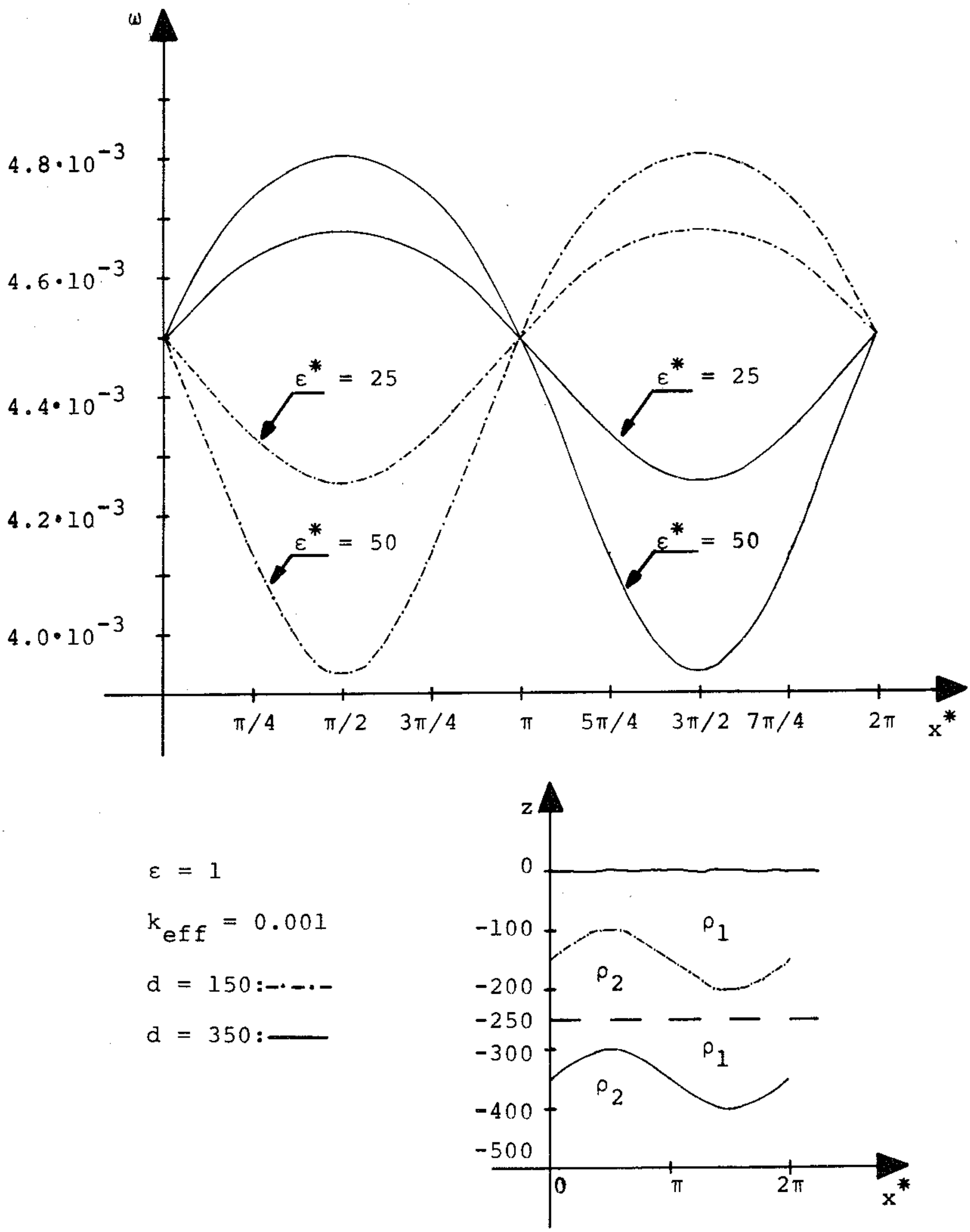

ω
4.8·10^-3
4.6·10^-3
4.4·10^-3
4.2·10^-3
4.0·10^-3
ε* = 25
ε* = 50
ε* = 25
ε* = 50
π/4
π/2
3π/4
π
5π/4
3π/2
7π/4
2π
x*
ε = 1
k_eff = 0.001
d = 150:
d = 350:
z
0
-100
-200
-250
-300
-400
-500
ρ1
ρ2
ρ1
ρ2
0
π
2π
x*

Fig. 25

4. INTERNAL WAVES IN A NON-HOMOGENEOUS VISCOUS MEDIUM

4.1 INTRODUCTION

In section 2.6 investigations of internal wave propagation in viscous non-homogeneous media were discussed. The investigations were based on a boundary layer type of approximation along the characteristics for the inviscid wave equation.

In this chapter the influence of viscosity on the propagation of internal waves generated by a harmonic disturbance travelling across a bottom covered by a non-homogeneous fluid is investigated. The approach to be used differs from the previously discussed methods by being based on the complete linear Boussinesq approximated equations of motion, and not on the boundary layer equations only.

It is assumed that the medium is static stable and that the fluid motion is two-dimensional. The coordinate system will have a horizontal x-axis and a vertical z-axis with z = 0 at the fluid surface. The depth of the fluid is D, see Fig. 26.

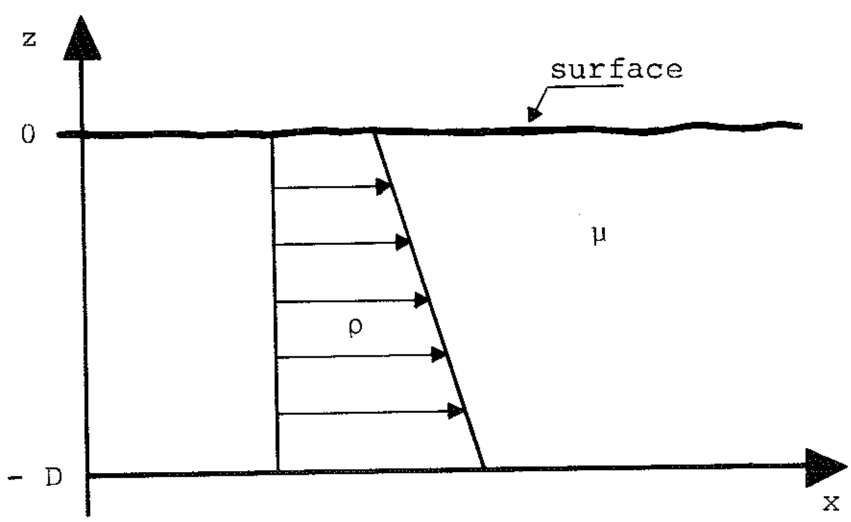


Fig. 26

4.2 THE WAVE EQUATION

The wave equation for internal gravity waves in a non-homogeneous viscous fluid will be derived from the Boussinesq approximated equation (see section 2.3) and the consequence of this approximation is, that the variations in the density are only taken into account in the buoyancy term $g(\rho - \rho_o)/\rho_o$ where ρ is the actual and ρ_o a reference density; in other terms of the equation of motion the density is assumed to have the constant value ρ_o.

In the equation of motion (2.3.13) the influence of the earth rotation is neglected ($\Omega_j \equiv 0$) and the viscosity ν ($\nu = \mu/\rho_o$) is assumed to be constant. This assumption together with the assumption that the salt diffusion effect might be neglected were discussed by MARTIN & LONG (1968) with special reference to the Boussinesq approximation.

Introducing the assumptions discussed leads to the following set of equations

$$\frac{\partial u_i}{\partial t} + u_j u_{i,j} = - \frac{1}{\rho_o} p_{,i} - g\delta_{i3} \frac{\rho - \rho_o}{\rho_o} + \nu u_{i,jj} \qquad (4.2.1)$$

$$u_{i,i} = 0 \qquad (4.2.2)$$

$$\frac{\partial \rho}{\partial t} + u_i \rho_{,i} = 0 \qquad (4.2.3)$$

The derivation of a single wave equation for the vertical velocity component $u_3 \equiv w$ follows the same principles as the previous derivations. It is again appropriate to introduce

$$\rho = \bar{\rho}(z) + \rho^*(x,z,t) \qquad (4.2.4)$$

and it should be noted that the mean density $\bar{\rho}$ is assumed to be a function of z only.

Rewriting equations (4.2.1), (4.2.2) and (4.2.3) gives

$$\frac{\partial u}{\partial t} + uu_{,x} + wu_{,z} = -\frac{1}{\rho_o} p_{,x} + \nu(u_{,xx} + w_{,zz}) \qquad (4.2.5)$$

$$\frac{\partial w}{\partial t} + uw_{,x} + ww_{,z} = -\frac{1}{\rho_o} p_{,z} - g\frac{\rho - \rho_o}{\rho_o} + \nu(w_{,xx} + w_{,zz}) \qquad (4.2.6)$$

$$u_{,x} + w_{,z} = 0 \qquad (4.2.7)$$

$$\frac{\partial \rho}{\partial t} + u\rho_{,x} + w\rho_{,z} = 0 \qquad (4.2.8)$$

The pressure p in the two components (4.2.5) and (4.2.6) of the equation of motion may be eliminated by differentiation of the u-equation with respect to z and of the w-equation with respect to x and by subtraction of the u-equation from the w-equation. These operations lead to

$$\frac{\partial}{\partial t} w_{,x} - \frac{\partial}{\partial t} u_{,z} + (uw_{,x})_{,x} - (uu_{,x})_{,z} + (ww_{,z})_{,x} - (wu_{,z})_{,z} = \nu[(\Delta w)_{,x} - (\Delta u)_{,z}] - g\left(\frac{\rho - \rho_o}{\rho_o}\right)_{,x} \qquad (4.2.9)$$

$$\frac{\partial}{\partial t} u_{,z} + (uu_{,x})_{,z} + (wu_{,z})_{,z} = -\frac{1}{\rho_o} p_{,xz} + \nu(\Delta u)_{,z} \qquad (4.2.10)$$

$$u_{,x} = - w_{,z} \qquad (4.2.11)$$

$$\frac{\partial \rho}{\partial t} = - u\rho_{,x} - w\rho_{,z} \qquad (4.2.12)$$

Differentiation of (4.2.9) with respect to x and t makes it possible to eliminate the $\partial^2 u_{,xz}/\partial t^2$ term and the $\partial(\Delta u)_{,xz}/\partial t$ term by use of (4.2.11) and to eliminate the $\partial\rho/\partial t$ term by use of equation (4.2.12) which gives

$$\frac{\partial^2}{\partial t^2}(w_{,xx} + w_{,zz}) + \frac{\partial}{\partial t}(uw_{,x})_{,xx} - \frac{\partial}{\partial t}(uu_{,x})_{,xz}$$

$$+ \frac{\partial}{\partial t}(ww_{,z})_{,xx} - \frac{\partial}{\partial t}(wu_{,z})_{,xz} =$$

$$\nu \frac{\partial}{\partial t}[(\Delta w)_{,xx} + (\Delta w)_{,zz}] + \frac{g}{\rho_o}(u\rho_{,x} + w\rho_{,z})_{,xx} \qquad (4.2.13)$$

Introduction of the assumption (4.2.4) leads to

$$\frac{\partial^2}{\partial t^2}(\Delta w) - \nu \frac{\partial}{\partial t}(\Delta\Delta w) - \frac{g}{\rho_o}[u\rho^*_{,x} + w(\bar{\rho}_{,z} + \rho^*_{,z})]_{,xx} =$$

$$\frac{\partial}{\partial t}[(wu_{,z})_{,xz} + (uu_{,x})_{,xz} - (uw_{,x})_{,xx} - (ww_{,z})_{,xx}] \qquad (4.2.14)$$

or

$$\frac{\partial^2}{\partial t^2}(\Delta w) - \nu \frac{\partial}{\partial t}(\Delta\Delta w) + N^2 w_{,xx} = \frac{g}{\rho_o}[u\rho^*_{,x} + w\rho^*_{,z}]_{,xx}$$

$$+ \frac{\partial}{\partial t}[(wu_{,z})_{,xz} + (uu_{,x})_{,xz} - (uw_{,x})_{,xx} - (ww_{,z})_{,xx}] \qquad (4.2.15)$$

where the Brunt-Väisälä frequency N was introduced.

Equation (4.2.15) is the complete wave equation for internal waves in a non-homogeneous viscous medium, for the description being based on the Boussinesq approximated equations of motion. The u-terms in the equation may be eliminated by using equation (4.2.11). For infinitesimal disturbances, the right hand side of equation (4.2.15) becomes negligible and (4.2.15) reduces to

$$\frac{\partial^2}{\partial t^2}\Delta w - \nu\frac{\partial}{\partial t}\Delta\Delta w + N^2 w_{,xx} = 0 \qquad (4.2.16)$$

It should be noted, that for inviscid flow, equation (4.2.16) is reduced to the well-known wave equation for internal gravity waves in a non-homogeneous fluid (2.6.33)

An important property of the wave equation for internal waves in an inviscid non-homogeneous fluid is that this equation is of the hyperbolic type if the frequency of the wave is less than the Brunt-Väisälä frequency, see section 2.6 for a discussion of the consequences of this property. From equation (4.2.16) it may be seen, that the wave equation for internal waves in a viscous non-homogeneous fluid is of the elliptic type. The type is determined† by the elliptic biharmonic operator.

4.3 THE SOLUTION OF THE WAVE EQUATION

Among the solutions to the wave equation (4.2.16), it is of special interest to investigate solutions which may be written as

$$w = \mathrm{Re}\{W(z)\ e^{i(kx-\omega t)}\} \tag{4.3.1}$$

i.e. solutions which are harmonic in the time and in the x-direction. The amplitude function W(z) specifies the velocity variations in the z-directions, and there are no a priori assumptions about W.

Introducing (4.3.1) into the wave equation (4.2.16) gives a fourth order ordinary differential equation for W. If the equation is reduced by the common factor $\exp\{i(kx - \omega t)\}$ and when prime denotes d()/dz one gets

$$-\omega^2(W'' - k^2W) + i\omega\nu(k^4W - 2k^2W'' + W'''') - N^2k^2W = 0 \tag{4.3.2}$$

which is rewritten

$$W'''' + W''\left\{\frac{i\omega}{\nu} - 2k^2\right\} + W\left\{k^4 + k^2\ \frac{i\omega}{\nu}\left(\frac{N^2}{\omega^2} - 1\right)\right\} = 0 \tag{4.3.3}$$

---

†See f.ex. WHITHAM (1974) for a discussion of the classification of partial differential equation.

Equation (4.3.3) is the differential equation from which W should be determined. The parameters ω, ν and k do not depend on z, while the Brunt-Väisälä frequency is in general a function of z being determined from equation (2.4.11) for a given density profile. In general one therefore has to resort to numerical solutions.

For the special case where N is a constant, i.e. where the density distribution is exponential and given by

$$\rho(z) = \rho_o \, e^{-\frac{N^2}{g} z} \tag{4.3.4}$$

equation (4.3.3) will have constant coefficients and for that case a complete analytical solution of (4.3.3) may be put down.

The complete solution of (4.3.3) for constant N is

$$W = A\,e^{R_1 z} + B\,e^{R_2 z} + C\,e^{R_3 z} + E\,e^{R_4 z} \tag{4.3.5}$$

where A, B, C and E are arbitrary constants and R_1, R_2, R_3 and R_4 are the roots of the equation

$$R^4 + \left(\frac{i\omega}{\nu} - 2k^2\right)R^2 + k^4 + k^2\,\frac{i\omega}{\nu}\left(\frac{N^2}{\omega^2} - 1\right) = 0 \tag{4.3.6}$$

given as

$$R_1 = \left[k^2 - \frac{1}{2}\,\frac{i\omega}{\nu} + i\sqrt{k^2\,\frac{i\omega}{\nu}\,\frac{N^2}{\omega^2} - \frac{1}{4}\left(\frac{i\omega}{\nu}\right)^2}\,\right]^{\frac{1}{2}} \tag{4.3.7}$$

$$R_2 = \left[k^2 - \frac{1}{2}\,\frac{i\omega}{\nu} - i\sqrt{k^2\,\frac{i\omega}{\nu}\,\frac{N^2}{\omega^2} - \frac{1}{4}\left(\frac{i\omega}{\nu}\right)^2}\,\right]^{\frac{1}{2}} \tag{4.3.8}$$

$$R_3 = -\,R_1 \tag{4.3.9}$$

$$R_4 = -\,R_2 \tag{4.3.10}$$

The result leads to a velocity distribution given by

$$w = \mathrm{Re}\left\{\left(A\, e^{R_1 z} + B\, e^{R_2 z} + C\, e^{-R_1 z} + E\, e^{-R_2 z}\right) e^{i(kx-\omega t)}\right\} \tag{4.3.11}$$

where R_1 and R_2 are determined by (4.3.7) and (4.3.8). The constants A, B, C and E must be determined from the actual boundary conditions.

4.4 THE WAVE FIELD ARISING FROM A HARMONIC DISTURBANCE TRAVELLING ACROSS THE BOTTOM

The solution (4.3.11) for the case of a wave field generated by a harmonic disturbance travelling across the bottom covered by a non-homogeneous fluid shall be investigated in what follows. But first the boundary conditions for this case should be stated. Within the Boussinesq approximation it is consistent to use

$$w(0) = 0 \tag{4.4.1}$$

i.e. the vertical velocity component is zero at the surface. If it is assumed, that the pressure above the free surface is constant and equal to the atmospheric pressure, the boundary condition for the normal stress at the surface will be

$$w_{,z} = 0 \quad \text{for} \quad z = 0 \tag{4.4.2}$$

The third boundary condition at the surface is, that the shear stress $\sigma_{xz} = \mu(w_{,x} + u_{,z})$ is zero, i.e.

$$u_{,z} + w_{,x} = 0 \quad \text{for} \quad z = 0 \tag{4.4.3}$$

The boundary condition at the bottom should describe the motion of a disturbance across the bottom in the z-direction, which is given by

$$w(-D) = w_o\, e^{i(k_o x - \omega_o t)} \tag{4.4.4}$$

where w_o is the velocity amplitude, k_o the wave number and ω_o the angular frequency of the wave travelling across the bottom.

The boundary conditions (4.4.1), (4.4.2), (4.4.3) and (4.4.4) give the four equations from which it is possible to determine the constants A, B, C and E to be inserted into the general solution (4.3.11).

From (4.4.3) it is seen that the horizontal velocity u enters one of the boundary conditions. The velocity u may be determined from equation (4.2.7) which together with (4.3.11) gives

$$u_{,x} = \left\{- AR_1\, e^{R_1 z} - BR_2\, e^{R_2 z} + CR_1\, e^{-R_1 z} + ER_2\, e^{-R_2 z}\right\} \times e^{i(kx-\omega t)} \quad (4.4.5)$$

which integrated leads to

$$u = f(z) + \left\{- AR_1\, e^{R_1 z} - BR_2\, e^{R_2 z} + CR_1\, e^{-R_1 z} + ER_2\, e^{R_2 z}\right\} \times \frac{e^{i(kx-\omega t)}}{ik} \quad (4.4.6)$$

where f(z) is an arbitrary function of z only.

Introduction of the velocities (4.3.11) and (4.4.6) into the boundary conditions now yields

$$A + B + C + E = 0 \quad (4.4.7)$$

$$AR_1 + BR_2 - CR_1 - ER_2 = 0 \quad (4.4.8)$$

$$f'(0) + \left\{- AR_1^2 - BR_2^2 - CR_1^2 - ER_2^2\right\} \frac{e^{i(kx-\omega t)}}{ik} + ik\left\{A + B + C + E\right\} e^{i(kx-\omega t)} = 0 \quad (4.4.9)$$

$$\left\{A\ e^{-R_1 D} + B\ e^{-R_2 D} + C\ e^{R_1 D} + E\ e^{R_2 D}\right\} e^{i(kx-\omega t)}$$

$$= w_o\ e^{i(k_o x-\omega_o t)} \qquad (4.4.10)$$

If (4.4.7) is introduced in (4.4.9) one gets the equation

$$ik\ f'(0) - \left\{AR_1^2 + BR_2^2 + CR_1^2 + ER_2^2\right\} e^{i(kx-\omega t)} = 0 \qquad (4.4.11)$$

which can only be valid for all values of x if

$$f'(0) = 0 \qquad (4.4.12)$$

and

$$AR_1^2 + BR_2^2 + CR_1^2 + ER_2^2 = 0 \qquad (4.4.13)$$

With the same argument, it is seen that equation (4.4.10) can only be valid if

$$k = k_o \qquad (4.4.14)$$

and

$$\omega = \omega_o \qquad (4.4.15)$$

(4.4.10) then becomes

$$A\ e^{-R_1 D} + B\ e^{-R_2 D} + C\ e^{R_1 D} + E\ e^{R_2 D} = w_o \qquad (4.4.16)$$

If the conditions (4.4.14) and (4.4.15) are inserted into the expressions for R_1 and R_2, equations (4.4.7), (4.4.8), (4.4.13) and (4.4.16) constitute the system from which A, B, C and E may be determined.

Written on matrix form, the linear system of equation (4.4.7), (4.4.8), (4.4.13) and (4.4.16) is

$$\underline{\underline{R}}\ \underline{A} = \underline{w} \qquad (4.4.17)$$

where the coefficient matrix $\underline{\underline{R}}$ and the vectors $\underline{A}$ and $\underline{w}$ are given as

$$\underline{\underline{R}} = \begin{Bmatrix} 1 & 1 & 1 & 1 \\ R_1 & R_2 & -R_1 & -R_2 \\ R_1^2 & R_2^2 & R_1^2 & R_2^2 \\ e^{-R_1 D} & e^{-R_2 D} & e^{R_1 D} & e^{R_2 D} \end{Bmatrix} \tag{4.4.18}$$

$$\underline{A} = \begin{Bmatrix} A \\ B \\ C \\ E \end{Bmatrix} \tag{4.4.19}$$

and

$$\underline{w} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ w_o \end{Bmatrix} \tag{4.4.20}$$

The manipulations necessary to solve the equations are shown in Appendix 4 and the result is

$$A = \frac{w_o R_2}{2} \left\{ R_1 \sinh(R_2 D) - R_2 \sinh(R_1 D) \right\}^{-1} \tag{4.4.21}$$

$$B = -\frac{w_o R_1}{2} \left\{ R_1 \sinh(R_2 D) - R_2 \sinh(R_1 D) \right\}^{-1} \tag{4.4.22}$$

$$C = -A \tag{4.4.23}$$

$$E = -B \tag{4.4.24}$$

Insertion of (4.4.21), (4.4.22), (4.4.23) and (4.4.24) into (4.3.11) noticing the equations (4.4.14) and (4.4.15) gives the solution for the vertical velocity distribution (see Appendix 4)

$$\frac{w}{w_o} = \mathrm{Re}\left\{ \frac{R_2 \ \sinh(R_1 z) \ - \ R_1 \ \sinh(R_2 z)}{R_1 \ \sinh(R_2 D) \ - \ R_2 \ \sinh(R_1 D)} \ e^{i(k_o x - \omega_o t)} \right\} \tag{4.4.25}$$

where the parameters R_1 and R_2 are given by

$$R_1 = \left[k_o^2 - \frac{1}{2}\,\frac{i\omega_o}{\nu} + i\sqrt{k_o^2\,\frac{i\omega_o}{\nu}\,\frac{N^2}{\omega^2} - \frac{1}{4}\left(\frac{i\omega_o}{\nu}\right)^2} \right]^{\frac{1}{2}} \tag{4.4.26}$$

and

$$R_2 = \left[k_o^2 - \frac{1}{2}\,\frac{i\omega_o}{\nu} - i\sqrt{k_o^2\,\frac{i\omega_o}{\nu}\,\frac{N^2}{\omega^2} - \frac{1}{4}\left(\frac{i\omega_o}{\nu}\right)^2} \right]^{\frac{1}{2}} \tag{4.4.27}$$

For a given physical system, i.e. for given values of the disturbance wave number k_o, the disturbance frequency ω_o, the fluid viscosity ν and the Brunt-Väisälä frequency N, one may calculate R_1 and R_2, and insertion of these values in (4.4.25) gives the distribution of the vertical velocity w as a function of the space coordinates x, z and the time t.

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APPENDIX 1

In this appendix an alternative method for the evaluation of the time derivative of the density term appearing in equation (3.2.9) is presented.

Assuming, that the variations in the pressure on a fluid element is mainly a result of the variations in its depth, one may write

$$dp = - g\rho\ dz \tag{A1.1}$$

If, as an approximation, the variations in pressure are assumed to take place isentropically, i.e. $dp = a^2\ d\rho$ where a is the sound speed, equation (A1.1) leads to†

$$\frac{d\rho}{dt} = - \frac{g\rho}{a^2}\ w \tag{A1.2}$$

or

$$\frac{d}{dt}\big((\rho - \rho_o)/\rho_o\big) = - \frac{g}{a^2}\frac{\rho}{\rho_o}\ w \simeq - \frac{g}{a^2}\ w \tag{A1.3}$$

Expansion of the substantial derivative gives

$$\frac{\partial}{\partial t}\big((\rho - \rho_o)/\rho_o\big) + u\big((\rho - \rho_o)/\rho_o\big)_{,x} + w\big((\rho - \rho_o)/\rho_o\big)_{,z} = - gw/a^2 \tag{A1.4}$$

and an application of the definition given in (3.2.11) finally leads to

$$g\ \frac{\partial}{\partial t}\big((\rho - \rho_o)/\rho_o\big) = - \frac{g}{\rho_o}\ u\bar{\rho}_{,x} - \frac{g}{\rho_o}\ w\bar{\rho}_{,z} - \frac{g}{\rho_o}\ u\rho^*_{,x} - \frac{g}{\rho_o}\ w\rho^*_{,z} - wg^2/a^2 \tag{A1.5}$$

† $\frac{d}{dt}$ is the substantial derivative.

The term wg^2/a^2 is negligible (for water $g^2/a^2 \simeq 4 \cdot 10^{-5}$) and thus it is seen, that inserting (A1.5), minus the term wg^2/a^2, into equation (3.2.9) again leads to (3.2.12) and, by use of the same arguments as in section 3.2, to equation (3.2.14).

APPENDIX 2

For the density distribution (3.3.10), the derivatives of the density given by equation (3.3.8) are listed below

$$\bar{\rho} = \rho_1 + \Delta\rho \cdot H(-d + \varepsilon^* \sin x^* - z) \quad \text{(A2.1)}$$

$$\frac{\partial \bar{\rho}}{\partial z} = -\Delta\rho \cdot \delta(-d + \varepsilon^* \sin x^* - z) \quad \text{(A2.2)}$$

$$\frac{\partial \bar{\rho}}{\partial x^*} = \Delta\rho \cdot \varepsilon^* \cdot \cos x^* \cdot \delta(-d + \varepsilon^* \sin x^* - z) \quad \text{(A2.3)}$$

$$\frac{\partial^2 \bar{\rho}}{\partial z \partial x^*} = -\Delta\rho \cdot \varepsilon^* \cdot \cos x^* \cdot \delta'(-d + \varepsilon^* \sin x^* - z) \quad \text{(A2.4)}$$

$$\frac{\partial^2 \bar{\rho}}{\partial x^{*2}} = \Delta\rho \cdot \varepsilon^* \big(-\sin x^* \cdot \delta(-d + \varepsilon^* \sin x^* - z) + \varepsilon^* \cos^2 x^* \cdot \delta'(-d + \varepsilon^* \sin x^* - z)\big) \quad \text{(A2.5)}$$

$$\frac{\partial^3 \bar{\rho}}{\partial x^{*2} \partial z} = \Delta\rho \cdot \varepsilon^* \big(\sin x^* \cdot \delta'(-d + \varepsilon^* \sin x^* - z) - \varepsilon^* \cdot \cos^2 x^* \cdot \delta''(-d + \varepsilon^* \sin x^* - z)\big) \quad \text{(A2.6)}$$

$$\frac{\partial^3 \bar{\rho}}{\partial x^{*3}} = \Delta\rho \cdot \varepsilon^* \big(-\cos x^* \cdot \delta(-d + \varepsilon^* \sin x^* - z) - 3\varepsilon^* \cdot \sin x^* \cos x^* \cdot \delta'(-d + \varepsilon^* \sin x^* - z) + \varepsilon^{*2} \cos^3 x^* \cdot \delta''(-d + \varepsilon^* \sin x^* - z)\big) \quad \text{(A2.7)}$$

APPENDIX 3

In this appendix the integration of the complete wave equation (3.3.16) across the density interface is carried out.

In order to stress the principles in the integration, only the important terms in the equation are written down, i.e. all constant factors will be omitted. In this sense we get

$$\int_{z_{disc-}}^{z_{disc+}} W'' \, dz + \int_{z_{disc-}}^{z_{disc+}} W'\left(\delta() + \delta'() + \delta''()\right) dz +$$

$$\int_{z_{disc-}}^{z_{disc+}} W\left(-(k + k^*)^2 + \delta() g \frac{\Delta\rho}{\rho_o} \frac{(k + k^*)^2}{\omega^2} + \delta'() + \delta''()\right) dz = 0 \tag{A3.1}$$

The individual terms are now integrated as follows:

The first integral is evaluated directly because W is a function of z only, i.e.

$$\int W'' \, dz = W' \tag{A3.2}$$

The second integral is integrated by parts leading to

$$\int W'\left(\delta() + \delta'() + \delta''()\right) dz =$$

$$\left[W\left(\delta() + \delta'() + \delta''()\right) - W\int\left(\delta'() + \delta''() + \delta'''()\right) dz\right]_{z_{disc-}}^{z_{disc+}} = 0 \tag{A3.3}$$

where it was utilized that W is continuous. The integration of the delta functions is easily performed by means of the formula

$$\int_{-\infty}^{\infty} f(\xi) \frac{\partial^n}{\partial x^n} \delta(\xi - x) d\xi = (-1)^n \frac{d^n}{dx^n} f(x) \tag{A3.4}$$

After these preliminary calculations, (A3.1) is easily integrated to give

$$W'(z_{disc+}) - W'(z_{disc-}) + W\left(-(k + k^*)^2(z_{disc+} - z_{disc-}) + g\,\frac{\Delta\rho}{\rho_o}\,\frac{(k + k^*)^2}{\omega^2}\right) = 0 \qquad (A3.5)$$

or

$$W'(z_{disc+}) - W'(z_{disc-}) = W(z_{disc})\left((k + k^*)^2\varepsilon - g\,\frac{\Delta\rho}{\rho_o}\,\frac{(k + k^*)^2}{\omega^2}\right) \qquad (A3.6)$$

Appendix 4

If the system of equations (4.4.7), (4.4.8), (4.4.13) and (4.4.16) are written in the matrix form (4.4.17), it may be seen that the solution is found by reduction of a matrix, which consists of $\underline{\underline{R}}$ and which has the vector $\underline{w}$ as a fifth column. This matrix is

$$\left\{\begin{matrix} 1 & 1 & 1 & 1 & 0 \\ R_1 & R_2 & -R_1 & -R_2 & 0 \\ R_1^2 & R_2^2 & R_1^2 & R_2^2 & 0 \\ e^{-R_1D} & e^{-R_2D} & e^{R_1D} & e^{R_2D} & w_o \end{matrix}\right\} \tag{A4.1}$$

Multiplying the first row by $-R_1$ and $-R_1^2$ and adding it to the second and third rows, respectively, lead to

$$\left\{\begin{matrix} 1 & 1 & 1 & 1 & 0 \\ 0 & R_2-R_1 & -2R_1 & -(R_1+R_2) & 0 \\ 0 & R_2^2-R_1^2 & 0 & R_2^2-R_1^2 & 0 \\ e^{-R_1D} & e^{-R_2D} & e^{R_1D} & e^{R_2D} & w_o \end{matrix}\right\} \tag{A4.2}$$

The third row is divided by the factor $R_2^2 - R_1^2$ and multiplied by -1 and then added to the first row and second it is divided by $-(R_2 - R_1)$ and then added to the second row, giving

$$\left\{\begin{matrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2R_1 & -2R_2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ e^{-R_1D} & e^{-R_2D} & e^{R_1D} & e^{R_2D} & w_o \end{matrix}\right\} \tag{A4.3}$$

Finally the first row is multiplied by $-e^{-R_2D}$ and is added to the fourth row, the second row is reduced by the factor -2 and the third row is multiplied by $-e^{R_2D}$ and is added to the fourth row which lead to

$$\begin{Bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & R_1 & R_2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2\sinh(R_1 D) & 2\sinh(R_2 D) & w_o \end{Bmatrix} \tag{A4.4}$$

From the first and the third row in (A4.4) is seen that

$$A = -C \tag{A4.5}$$

and

$$B = -E \tag{A4.6}$$

The second row in (A4.4) gives

$$C = -\frac{R_2}{R_1} E \tag{A4.7}$$

and the fourth row then gives

$$E = \frac{w_o R_1}{2\{R_1 \sinh(R_2 D) - R_2 \sinh(R_1 D)\}} \tag{A4.8}$$

Introducing this in (A4.7) leads to

$$C = \frac{-w_o R_2}{2\{R_1 \sinh(R_2 D) - R_2 \sinh(R_1 D)\}} \tag{A4.9}$$

The equations (A4.5), (A4.6), (A4.8) and (A4.9) gives the values of the constants A, B, C and E for the boundary conditions described in section (4.4) and by insertion of these constants into the general solution (4.3.11) the following expression for w is obtained

$$w = \mathrm{Re}\left\{\frac{w_o}{R_1 \sinh(R_2 D) - R_2 \sinh(R_1 D)} \times \right.$$

$$\left.\left[\frac{R_2}{2} e^{R_1 z} - \frac{R_1}{2} e^{R_2 z} - \frac{R_2}{2} e^{-R_1 z} + \frac{R_1}{2} e^{-R_2 z}\right] e^{i(k_o x - \omega_o t)}\right\} \quad (A4.10)$$

or alternatively

$$\frac{w}{w_o} = \mathrm{Re}\left\{\frac{R_2 \sinh(R_1 z) - R_1 \sinh(R_2 z)}{R_1 \sinh(R_2 D) - R_2 \sinh(R_1 D)} e^{i(k_o x - \omega_o t)}\right\} \quad (A4.11)$$

Mogens Terp Paulsen: Internal Waves in Nonhomogeneous Media.

ERRATA.

p. 4_5:

P Pressure should read p Pressure

p. 19, (2.4.15):

$\frac{\partial^2 \xi}{\partial t^2} - N^2 \xi = 0$ should read $\frac{\partial^2 \xi}{\partial t^2} + N^2 \xi = 0$

p, 33, (2.6.2):

$\rho(z) =$ should read $\rho(z + \xi) =$

p. 44, (2.6.57a):

ℓ should read 1

p. 69, (4.2.5):

$w_{,zz}$ should read $u_{,zz}$

P. 77, (4.4.26) and (4.4.27):

ω should read ω_o